

VERMA-TYPE OBJECTS IN CATEGORY $O_{c,\nu}$

INNA ENTOVA AIZENBUD

ABSTRACT. We study a family of abelian categories $O_{c,\nu}$ depending on complex parameters c, ν which are interpolations of the O -category for the rational Cherednik algebra $H_c(\nu)$ of type A , where ν is a positive integer. We define the notion of a Verma object in such a category (a natural analogue of the notion of Verma module).

We give some necessary conditions and some sufficient conditions for the existence of a non-trivial morphism between two such Verma objects. We also compute the character of the irreducible quotient of a Verma object for sufficiently generic values of parameters c, ν , and prove that a Verma object of infinite length exists in $O_{c,\nu}$ only if $c \in \mathbb{Q}_{<0}$. We also show that for every $c \in \mathbb{Q}_{<0}$ there exists $\nu \in \mathbb{Q}_{<0}$ such that there exists a Verma object of infinite length in $O_{c,\nu}$.

The latter result is an example of a degeneration phenomenon which can occur in rational values of ν , as was conjectured by P. Etingof.

1. INTRODUCTION

The study of representations in complex rank involves defining and studying families of abelian categories depending on a parameter t which are polynomial interpolations of the categories of representations of objects such as finite groups, Lie groups, Lie algebras and more. This was done by P. Deligne in [5] for finite dimensional representations of the general linear group GL_n , the orthogonal and symplectic groups O_n, Sp_{2n} and the symmetric group S_n . Deligne defined Karoubian tensor categories $Rep(GL_t), Rep(OSp_t), Rep(S_t)$, $t \in \mathbb{C}$, which in points $n = t \in \mathbb{Z}_+$ allow an essentially surjective functor onto the standard categories $Rep(GL_n), Rep(OSp_n), Rep(S_n)$. The category $Rep(S_t)$ was subsequently studied by himself and others (F. Knop in [15], V. Ostrik, J. Comes in [4]).

The resulting categories can be shown to have superexponential growth, and are not equivalent to the category of finite-dimensional representations of any affine algebraic group or supergroup. One can look for rational non-integer values of t where degeneration occurs, which would allow us to understand better how the structure of the classical category of representations is similar for all ranks n , and why degenerate phenomena occur at specific values of n .

P. Etingof, in [7], suggested using these results to define and study the categories of representations of rational Cherednik algebras, as well as Lie superalgebras, affine Lie algebras and other “non-compact” representation theory objects. A possible approach, as stated in [7], would be using the tensor categories $Rep(GL_t), Rep(OSp_t), Rep(S_t)$ and defining the required category for a “non-compact” object in complex rank as a category of tuples which consist of an ind-object of the corresponding tensor category along with morphisms satisfying some relations.

In this paper, we study the categories which are interpolations of the classical O -category of lowest weight modules for the rational Cherednik algebra of type A , denoted by $H_c(n)$. These categories are BGG-type categories, where one can define Verma objects and study morphisms between these, as well as their irreducible quotients.

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The categories $\text{Rep}(H_c(\nu))$ (the parameter ν replacing the parameter t used above) have been defined in [7] as categories whose objects are ind-objects of Deligne's category $\text{Rep}(S_\nu)$ with additional structure; namely, two morphisms in $\text{Rep}(S_\nu)$, denoted by x, y , satisfying some relations. These morphisms represent the actions of the elements x_1, \dots, x_n and y_1, \dots, y_n of $H_c(n)$ respectively. We proceed to define the full subcategory $O_{c,\nu}$ of $\text{Rep}(H_c(\nu))$, which is an interpolation of the category O of the classical Cherednik algebra $H_c(n)$.

In this category $O_{c,\nu}$ we have objects we will call “Verma objects”, as they are analogues of the Verma modules in the classical O category. They are parameterized by arbitrary partitions λ , and denoted by $M_{c,\nu}(\lambda)$. As in the $H_c(n)$ case, they have a natural grading as $\text{Rep}(S_\nu)$ ind-objects, with the lowest grade consisting of an indecomposable $\text{Rep}(S_\nu)$ -object given by the partition λ .

The Verma objects have about the same properties as the classical Verma modules, and one can ask for which values of parameters c, ν they are reducible, and for which values of parameters c, ν there are non-trivial morphisms between different Verma objects. It would also be interesting to compute the character of the unique irreducible quotient of $M(\lambda)$ (the character being its decomposition as a $\text{Rep}(S_\nu)$ ind-object into a sum of irreducible $\text{Rep}(S_\nu)$ -objects).

1.1. Summary of the results. Recall that the category $\text{Rep}(S_\nu)$ is abelian semisimple whenever $\nu \notin \mathbb{Z}_+$, and for any ν , its indecomposable objects are parameterized by Young diagrams of arbitrary size. One can treat an indecomposable object X_τ of $\text{Rep}(S_\nu)$, with τ being a Young diagram, as corresponding, in fact, to the Young diagram obtained from τ by adding a very long top row (“of size $\nu - |\tau|$ ”).

Throughout the paper we will assume that $\nu \notin \mathbb{Z}_+$ unless it is explicitly stated otherwise.

The indecomposable object in $\text{Rep}(S_\nu)$ corresponding to a Young diagram with one box is denoted by \mathfrak{h}_0 , and has dimension $\nu - 1$ (it corresponds to the reflection representation of S_n when $\nu = n$). One can define an algebra object $S\mathfrak{h}_0^*$ in $\text{ind-Rep}(S_\nu)$ (which would correspond to the ring $\mathbb{C}[x_1, \dots, x_n]/\langle \mathbb{C}[x_1, \dots, x_n] (\sum_i x_i) \rangle$ when $\nu = n$).

We define the Verma objects $M_{c,\nu}(\lambda)$ of $O_{c,\nu}$ in Subsection 6.2. The underlying ind-object of $\text{Rep}(S_\nu)$ is just $S\mathfrak{h}_0^* \otimes \lambda$.

Notation 1.1.1. • Denote the set of points (c, ν) on the complex plane where $M_{c,\nu}(\tau)$ is reducible by B_τ . The image of $B_\tau \setminus \{c = 0\}$ under the map $c \mapsto 1/c$ will be denoted by \mathcal{B}_τ .

- Denote the set of points (c, ν) on the complex plane where there is a non-zero morphism $M_{c,\nu}(\mu) \longrightarrow M_{c,\nu}(\tau)$ by $B_{\mu,\tau}$. The image of $B_{\mu,\tau} \setminus \{c = 0\}$ under the map $c \mapsto 1/c$ will be denoted by $\mathcal{B}_{\mu,\tau}$.

As in the classical case, we have: $B_\tau = \bigcup_\mu B_{\mu,\tau}$, the parameter μ running over all Young diagrams of arbitrary size.

We will show that if $\mu \neq \tau$, $B_{\mu,\tau}$ lies inside a countable (disjoint) union of curves $\bigcup_{m \in \mathbb{Z}_{>0}} L_{\tau,\mu,m}$ for which we give explicit equations. These curves become straight lines, denoted by $\mathcal{L}_{\tau,\mu,m}$, when we switch to parameters $(1/c, \nu)$ instead of (c, ν) .

For a fixed positive integer m , all points (c, ν) which satisfy the following condition lie on the curve $L_{\tau,\mu,m}$:

Condition 1.1.2. There exists a non-zero map $M_{c,\nu}(\mu) \longrightarrow M_{c,\nu}(\tau)$ with the lowest grade of $M(\mu)$ mapping to grade m of $M(\tau)$

The equations for lines $\mathcal{L}_{\tau,\mu,m}$ are given in Section 7.

It is easy to see that the intersection of $\mathcal{B}_{\mu,\tau}$ with each of these lines $\mathcal{L}_{\tau,\mu,m}$ is either the entire line or a finite number of points. We give a full description of the lines $\mathcal{L}_{\tau,\mu,m}$ lying inside $\mathcal{B}_{\mu,\tau}$ whenever $|\tau| \neq |\mu|$ in Section 10:

Theorem 1. *For two Young diagrams μ, τ and an integer $m > 0$, the following are equivalent:*

- (1) $|\mu| \neq |\tau|$ and $\mathcal{L}_{\tau,\mu,m} \subset \mathcal{B}_{\mu,\tau}$,
- (2) $\mu = \mathbf{rec}(\tilde{\tau}_{j_s} + \text{sign}(|\mu| - |\tau|), \mathbf{core}_{(\nu-s)}(\tau))$ for some $s \in C_\tau$, and j_s given by $s = |\tau| - 1 + j_s - \tilde{\tau}_{j_s}$.

Here $\mathbf{rec}(\tilde{\tau}_{j_s} + \text{sign}(|\mu| - |\tau|), \mathbf{core}_{(\nu-s)}(\tau))$ is a construction described explicitly in Section 9, and $C_\tau = \{|\tau| - 1 + j - \tilde{\tau}_j \mid j \in \mathbb{Z}_{>0}\}$.

In terms of Young diagrams with “very long top row”, this construction means that

- we start with the Young diagram obtained from τ by adding a very long top row (“of size $\nu - |\tau|$ ”),
- take out a hook of size $\nu - s$ with leg $\tilde{\tau}_{j_s} + \text{sign}(|\mu| - |\tau|)$ (the Young diagram obtained at this stage is denoted by $\mathbf{core}_{(\nu-s)}(\tau)$, and is a “normal” Young diagram, i.e. all its rows are of integer size),
- insert a hook of size $\nu - s$ with leg $\tilde{\tau}_{j_s} + \text{sign}(|\mu| - |\tau|) + 1$.

The diagram obtained this way would be the Young diagram obtained from μ by adding a very long top row (“of size $\nu - |\mu|$ ”), where $\mu = \mathbf{rec}(\tilde{\tau}_{j_s} + \text{sign}(|\mu| - |\tau|), \mathbf{core}_{(\nu-s)}(\tau))$.

From the theorem above one can easily see that if $|\mu| \neq |\tau|$, and $\mathcal{L}_{\tau,\mu,m} \subset \mathcal{B}_{\mu,\tau}$, then for a generic point $(\frac{1}{c}, \nu)$ of $\mathcal{L}_{\tau,\mu,m}$, $(\frac{1}{c}, \nu) \notin \mathcal{B}_{\mu',\tau}$ for any $\mu' \neq \mu$ (see Proposition 10.3).

Furthermore, we give a formula for the character of $L_{c,\nu}(\tau)$ when $(1/c, \nu)$ is a generic point on a line $\mathcal{L}_{\tau,\mu,m} \subset \mathcal{B}_{\mu,\tau}$, $|\tau| \neq |\mu|$, and compute explicitly the character of $L_{c,\nu}(\emptyset)$, where $(1/c, \nu)$ is a generic point on a line $\mathcal{L}_{\emptyset,\mu,k} = \{(1/c, \nu) \mid c\nu = k\}$ (here $k \in \mathbb{Z}_{>0}$ is fixed).

We have proved that in such a case $\mu = \mathbf{rec}(\tilde{\tau}_{j_s} + \text{sign}(|\mu| - |\tau|), \mathbf{core}_{(\nu-s)}(\tau))$ for some $s \in C_\tau$, and j_s given by $s = |\tau| - 1 + j_s - \tilde{\tau}_{j_s}$.

In that case there exists a long exact sequence

$$\begin{aligned} \dots &\longrightarrow M_{c,\nu}(\mathbf{rec}(\tilde{\tau}_{j_s} \pm l, \mathbf{core}_{(\nu-s)}(\tau))) \longrightarrow M_{c,\nu}(\mathbf{rec}(\tilde{\tau}_{j_s} \pm (l-1), \mathbf{core}_{(\nu-s)}(\tau))) \longrightarrow \dots \\ \dots &\longrightarrow M_{c,\nu}(\mathbf{rec}(\tilde{\tau}_{j_s} \pm 1, \mathbf{core}_{(\nu-s)}(\tau))) \longrightarrow M_{c,\nu}(\tau) \longrightarrow L_{c,\nu}(\tau) \longrightarrow 0 \end{aligned}$$

The sign is $\text{sign}(|\mu| - |\tau|)$, and this sequence ends with $M(\mathbf{rec}(0, \mathbf{core}_{(\nu-s)}(\tau)))$ (on the left) if $|\mu| - |\tau| < 0$.

The character formula for $L_{c,\nu}(\tau)$ is then obtained from Euler’s formula applied to the above long exact sequence.

We conclude the paper by proving the following theorem:

Theorem 2. *The only values of c for which Verma objects of infinite length can occur are $c \in \mathbb{Q}_{<0}$.*

This is, in fact, a sufficient condition (on c) for the existence of a Verma object of infinite length: one can show that for any $c \in \mathbb{Q}_{<0}$ there exists $\nu \in \mathbb{Q}_{<0}$ such that $M_{c,\nu}(\emptyset)$ has infinite length.

Theorem 2 is an example of a degenerate phenomenon which can occur for rational non-integer values of ν . The classical representation theory of Cherednik algebras says that for a non-negative integer n , all modules in $O(H_c(n))$ have finite length (cf. [9, Corollary 3.26]), and we see from Theorem 2 that the same is true for Verma objects in $O_{c,\nu}$ for $\nu \notin \mathbb{Q}$.

It would be interesting to know about other phenomena which can occur in $O_{c,\nu}$ only for $\nu \in \mathbb{Q} \setminus \mathbb{Z}_+$. Such phenomena were conjectured by P. Etingof in [7].

1.2. Structure of the paper. In Section 3, we recall basic definitions and facts about $H_c(n)$, the Cherednik algebra of type A . Some further facts about the blocks of the O -category for $H_c(n)$ will be given in 8.

In Section 4 we recall basic facts about the Deligne category $\text{Rep}(S_\nu)$. We do not give a definition (it can be found in [5]), instead we mention the facts about this category which we will use.

In Sections 5 and 6, we define the category $\text{Rep}(H_c(\nu))$ (interpolation of the category of representations of $H_c(n)$) and the category $O_{c,\nu}$ (interpolation of the O -category of $H_c(n)$). We then define the Verma objects $M_{c,\nu}(\tau)$ in $O_{c,\nu}$.

In Section 7, we give a necessary condition on (c, ν) which is required in order for a non-trivial map $M_{c,\nu}(\mu) \rightarrow M_{c,\nu}(\tau)$ to exist. This condition is an equation which defines the lines $\mathcal{L}_{\tau,\mu,m}$.

In Section 9 we define constructions for $O_{c,\nu}$ which are necessary in order to describe the triplets (τ, μ, m) for which $\mathcal{L}_{\tau,\mu,m} \subset \mathcal{B}_{\mu,\tau}$ (the latter is done in Section 10; we give a full description in the case $|\tau| \neq |\mu|$, and a partial description when $|\tau| = |\mu|$).

In Section 11 we define the formal character of an object of $O_{c,\nu}$, give a formula for the character of a Verma object $M_{c,\nu}(\tau)$, and compute the character of a simple object $L_{c,\nu}(\tau)$ in the simplest cases. We also give a positive character formula for the Verma objects and look at examples when such a formula can be derived for some simple objects $L_{c,\nu}(\tau)$.

In Section 12 we prove Theorem 2 mentioned above, and in particular, give a lower bound on the grade of a given Verma object $M_{c,\nu}(\tau)$ in which a simple $\text{Rep}(S_\nu)$ -object X_μ can lie.

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2. NOTATIONS AND CONVENTIONS

The base field throughout this paper will be \mathbb{C} .

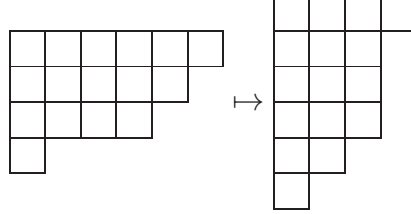
Notation 2.0.1. For a rational number $q \in \mathbb{Q}$, we write $q = \frac{\text{num}(q)}{\text{den}(q)}$, where $\text{num}(q), \text{den}(q) \in \mathbb{Z}, \text{den}(q) > 0, \text{gcd}(\text{num}(q), \text{den}(q)) = 1$.

Notation 2.0.2 (Symmetric group and Young diagrams).

- S_n will denote the symmetric group ($n \in \mathbb{Z}_+$). We will denote by S the set of reflections s_{ij} in S_n .
- The notation λ will stand for a partition (weakly decreasing sequence of non-negative integers), a Young diagram λ (considered in the English notation), and the corresponding irreducible representation of $S_{|\lambda|}$. Here $|\lambda|$ is the sum of entries of the partition, or, equivalently, the number of cells in the Young diagram λ .
- When referring to a cell in a Young diagram λ , (i, j) will be the cell in row i and column j , with $i, j \geq 1$.
- The length of the partition λ , i.e. the number of rows of Young diagram λ , will be denoted by $l(\lambda)$.

- The i -th entry of a partition λ , as well as the length of the i -th row of the corresponding Young diagram, will be denoted by λ_i (if $i > l(\lambda)$, then $\lambda_i := 0$). The transpose of the Young diagram λ will be denoted by λ' (so the length of the i -th column of λ is λ'_i).

Example 2.0.3. Consider the Young diagram λ corresponding to the partition $(6, 5, 4, 1)$. The length of λ is 4, the size is 16, and the transpose of λ is the young diagram corresponding to partition $(4, 3, 3, 3, 2, 1)$:



- \mathfrak{h} (in context of representations of S_n) will denote the permutation representation of S_n , i.e. the n -dimensional representation \mathbb{C}^n with S_n acting by $g.e_j = e_{g(j)}$ on the standard basis e_1, \dots, e_n of \mathbb{C}^n .
- \mathfrak{h}_0 (in context of representations of S_n) will denote the reflection representation of the symmetric group S_n : this is the $n - 1$ dimensional irreducible representation of S_n corresponding to the partition $(n - 1, 1)$ of n . This representation is the subrepresentation of \mathfrak{h} given by the S_n -invariant subspace of all vectors whose sum of coordinates is 0.

We will use the same notation for the corresponding object in $Rep(S_\nu)$.

- Let $s \in \mathbb{Z}_{\geq -1}$, $k \in \mathbb{Z}_{>0}$. We will denote by τ^s a Young diagram consisting of a row with $s + 1$ cells, and by π^k a Young diagram consisting of a column with k cells.

Definition 2.0.4 (Block of an abelian category). A block in an abelian category is a full subcategory generated by an equivalence class of indecomposable objects, defined by the following equivalence relation:

This relation is the minimal equivalence relation such that any two indecomposable objects with a non-zero morphism between them are equivalent.

3. CLASSICAL CHEREDNIK ALGEBRA OF TYPE A

This section follows [9, chapter 2], [10, section 2].

3.1. Rational Cherednik algebra of type A. Let $c \in \mathbb{C}$ (can be considered as a formal parameter).

Definition 3.1.1. The rational Cherednik algebra of type A of rank n with parameter c , denoted by $\overline{H}_c(n)$, is the quotient of the algebra $\mathbb{C}S_n \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the ideal generated by the following relations:

- (1) $[x, x'] = 0, [y, y'] = 0$ for any $x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h}$,
- (2) $[y, x] = (y, x) - \frac{c}{2} \sum_{s \in S} ((1-s)y, (1-s)x)s$, where $x \in \mathfrak{h}^*, y \in \mathfrak{h}$, $(,)$ is the natural pairing between $\mathfrak{h}, \mathfrak{h}^*$.

If we choose dual bases x_1, \dots, x_n of \mathfrak{h}^* , y_1, \dots, y_n of \mathfrak{h} such that $\forall i, j, (y_j, x_i) = \delta_{ij}$, and $s_{ij}(y_i) = y_j, s_{ij}(x_i) = x_j$, then the last relation from Definition 3.1.1 can be rewritten as

$$[y_j, x_i] = cs_{ij} \text{ for any } i \neq j, [y_i, x_i] = 1 - c \sum_{j \neq i} s_{ij}$$

Similarly to the universal enveloping algebra of a Lie algebra, we have:

Proposition 3.1.2 (PBW-type theorem). $\overline{H}_c(n) \cong S\mathfrak{h} \otimes \mathbb{C}S_n \otimes S\mathfrak{h}$ as vector spaces.

Continuing the analogy between universal enveloping algebra of a Lie algebra and a Cherednik algebra, we have a Cartan-type element:

Definition 3.1.3. The element $\overline{\mathbf{h}} \in \overline{H}_c(n)$ is defined as

$$\overline{\mathbf{h}} := \sum x_i y_i + \dim(\mathfrak{h})/2 - c \sum_{s \in S} s = \sum x_i y_i + n/2 - c \sum_{s \in S} s$$

This element satisfies: $g\overline{\mathbf{h}}g^{-1} = \overline{\mathbf{h}}, [\overline{\mathbf{h}}, x_i] = x_i, [\overline{\mathbf{h}}, y_i] = -y_i$ for any $g \in S_n, i, j \in \{1, \dots, n\}$, and can also be written as $\overline{\mathbf{h}} = \frac{1}{2} \sum_i (x_i y_i + y_i x_i)$.

We will use the “reduced” Cherednik algebra $H_c(n)$, which corresponds to the reflection representation $\mathfrak{h}_0 = \text{span}_{\mathbb{C}}\{y_i - y_j \mid i, j \in \{1, \dots, n\}, i \neq j\}$ rather than the permutation representation \mathfrak{h} of S_n . It is defined in the same way as $\overline{H}_c(n)$, but with \mathfrak{h}_0 appearing instead of \mathfrak{h} (see [9], or [13][Section 2]). In fact, we have: $\overline{H}_c(n) = H_c(n) \otimes \mathbb{A}_1$, \mathbb{A}_1 being the Weyl algebra. This follows from the fact that the subalgebra of $\overline{H}_c(n)$ generated by $\sum y_i, \sum x_i$ is isomorphic to \mathbb{A}_1 , and commutes with the subalgebra $H_c(n)$ of $\overline{H}_c(n)$.

For $H_c(n)$, $\overline{\mathbf{h}}$ is not an element of $H_c(n)$, but the element

$$\mathbf{h} := \overline{\mathbf{h}} - \frac{(\sum x_i)(\sum y_i) + (\sum y_i)(\sum x_i)}{2n}$$

of $H_c(n)$ plays the analogous role.

3.2. Category $O(H_c(n))$. The category $O(H_c(n))$ is defined as the category of all modules over $H_c(n)$ which are finitely generated over $S\mathfrak{h}_0^*$, and on which \mathfrak{h}_0 acts locally nilpotently.

3.3. Verma modules. Let τ be an irreducible representation of S_n . Then one can define $M_{c,n}(\tau) := H_c(n) \otimes_{\mathbb{C}S_n \ltimes S\mathfrak{h}_0} \tau$, where $S\mathfrak{h}_0$ acts on τ by 0. This is the Verma module corresponding to τ . It is a lowest weight module whose underlying space is isomorphic to $\tau \otimes S\mathfrak{h}_0^*$, where S_n acts on each subspace $\tau \otimes S^m \mathfrak{h}_0^*$, \mathfrak{h}_0^* acts by multiplication on the right, and \mathfrak{h}_0 acts according to the commutator relation given in the definition (if $c = 0$, then \mathfrak{h}_0 acts by “evaluation” on the right). The element \mathbf{h} acts locally finitely on modules from $O(H_c(n))$, with finite dimensional generalized eigenspaces. In particular, \mathbf{h} acts semisimply on $M_{c,n}(\tau)$, with lowest eigenvalue

$$h_{c,n}(\tau) = \dim(\mathfrak{h}_0)/2 - c \sum_{s \in S} s|_{\tau}$$

(the rest of the eigenvalues on $\tau \otimes S^m \mathfrak{h}_0^*$, $m > 0$ being $h_{c,n}(\tau) + m$).

3.3.1. Jucys-Murphy element. Recall that the Jucys-Murphy central element for S_n is $\Omega := \sum_{s \in S} s \in \mathbb{C}[S_n]$. This element acts on an irreducible representation parameterized by the Young diagram μ by the scalar $ct(\tau) = \sum_{(i,j) \in \tau} (j - i)$, called “content of τ ” (here (i, j) denotes cell in row i , column j of the Young diagram τ).

So $h_{c,n}(\tau) = \dim(\mathfrak{h}_0)/2 - c \sum_{s \in S} s|_{\tau} = \dim(\mathfrak{h}_0)/2 - c\Omega|_{\tau} = \frac{n-1}{2} - c \cdot ct(\tau)$.

4. CATEGORY $\text{Rep}(S_{\nu})$

4.1. General description. We recall some basic facts about Deligne’s category $\text{Rep}(S_{\nu})$ (see [5], [7], and [19, Section 2]).

This is a family of Karoubian tensor categories over \mathbb{C} defined for any $\nu \in \mathbb{C}$, and “flat with respect to parameter ν ” (one can view ν as a formal parameter, with $\text{Obj}(\text{Rep}(S_{\nu}))$ not depending on ν , and the Hom spaces being modules over $\mathbb{C}[\nu]$).

For $\nu \notin \mathbb{Z}_+$, $\text{Rep}(S_\nu)$ is a semisimple abelian category. But if ν is a non-negative integer, then the category $\text{Rep}(S_\nu)$ has a tensor ideal \mathfrak{I}_ν (ideal of morphisms $f : X \rightarrow Y$ such that $\text{tr}(fu) = 0$ for any morphism $u : Y \rightarrow X$), and the classical category $\text{Rep}(S_n)$ of finite-dimensional representations of the symmetric group for $n := \nu$ is equivalent to $\text{Rep}(S_\nu)/\mathfrak{I}_\nu$.

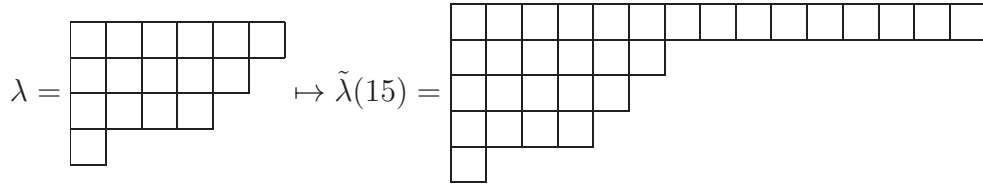
Notation 4.1.1. We will denote Deligne's category for integer value $n \geq 0$ of ν as $\text{Rep}(S_{\nu=n})$, to distinguish it from the classical category $\text{Rep}(S_n)$ of representations of the symmetric group S_n . Similarly for other categories arising in this text.

The indecomposable objects of $\text{Rep}(S_\nu)$, regardless of the value of ν , are labeled by all Young diagrams (of arbitrary size). We will denote the indecomposable object in $\text{Rep}(S_\nu)$ corresponding to the Young diagram τ by X_τ .

For non-negative integer $\nu =: n$, we have: the partitions λ for which X_λ has a non-zero image in the quotient $\text{Rep}(S_\nu)/\mathfrak{I}_\nu \cong \text{Rep}(S_n)$ are exactly the λ for which $\lambda_1 + |\lambda| \leq n$.

Notation 4.1.2. We will denote by $\tilde{\lambda}(n)$ the Young diagram obtained from λ by adding a top row of size $n - |\lambda|$, where $n \geq \lambda_1 + |\lambda|$ is an integer.

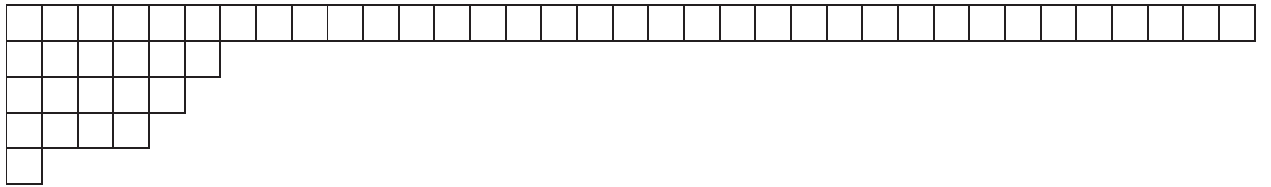
Example 4.1.3.



If $\lambda_1 + |\lambda| \leq n$, then the image of λ in $\text{Rep}(S_n)$ is the irreducible representation of S_n corresponding to the Young diagram $\tilde{\lambda}(n)$.

Intuitively, one can treat the indecomposable objects of $\text{Rep}(S_\nu)$ as if they were parametrized by “Young diagrams with very long top row”. The indecomposable object X_λ would be treated as if it corresponded to $\tilde{\lambda}(\nu)$, i.e. a Young diagram obtained by adding a very long top row (“of size $\nu - |\lambda|$ ”). This point of view is useful to understand how to extend constructions for S_n involving Young diagrams to $\text{Rep}(S_\nu)$.

Example 4.1.4. The simple object X_λ , where $\lambda = \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \\ \hline \square & \square & \square & \square & \\ \hline \square & \square & \square & \square & \\ \hline \square & & & & \\ \hline \end{array}$ can be thought of as a Young diagram with a “very long top row of length $(\nu - 16)$ ”:



Notation 4.1.5. Let \mathfrak{h}_0 be the irreducible object in $\text{Rep}(S_\nu)$ corresponding to the one-box Young diagram (that would be the analogue of the reflection representation in $\text{Rep}(S_n)$).

Its dual in $\text{Rep}(S_\nu)$, denoted by \mathfrak{h}_0^* , is isomorphic to \mathfrak{h}_0 , since all the objects in $\text{Rep}(S_\nu)$ have a symmetric form defined on them (analogue of the invariant symmetric form on the representations of the symmetric group over a field of characteristic zero).

4.2. Pieri's rule. We have an interpolation of Pieri's rule (see [4, Proposition 5.15], [7], and [11] for Pieri's rule in the integer ν case):

Proposition 4.2.1. [*Pieri's rule*] Let X_τ be a simple object of $\text{Rep}(S_\nu)$. As objects of $\text{Rep}(S_\nu)$,

$$\mathfrak{h}_0 \otimes X_\tau \cong \bigoplus_{\mu \in P_\tau^+ \cup P_\tau^- \cup P_\tau^0} X_\mu \oplus cc(\tau)X_\tau$$

where $P_\tau^+, P_\tau^-, P_\tau^0$ are the sets of all Young diagrams obtained from τ by adding, deleting, or moving a corner cell, respectively (note that $\tau \notin P_\tau^0$), and $cc(\tau)$ is the number of corner cells of τ .

Example 4.2.2. $X_{\pi^k} \cong \Lambda^k \mathfrak{h}_0$ as objects of $\text{Rep}(S_\nu)$.

4.3. Jucys-Murphy element for $\text{Rep}(S_\nu)$. One can define an analogue of the Jucys-Murphy element for a simple object in $\text{Rep}(S_\nu)$ parameterized by a Young diagram μ :

Definition 4.3.1. Given a simple object X_μ in $\text{Rep}(S_\nu)$, one can define an endomorphism Ω of X_μ :

$$\Omega|_{X_\mu} := \frac{(\nu - |\mu|)(\nu - |\mu| - 1)}{2} - |\mu| + ct(\mu)$$

So if $\text{Rep}(S_\nu)$ is semisimple, one can define the generalized Jucys-Murphy element as an element of $Z(\text{Rep}(S_\nu))$ (center of Deligne's category). In fact, J. Comes and V. Ostrik in [4] defined an element Ω of $Z(\text{Rep}(S_\nu))$ for arbitrary ν (including non-negative integers) coinciding with the element defined above for $\nu \notin \mathbb{Z}_+$.

5. CATEGORY $\text{Rep}(H_c(\nu))$

Let $c, \nu \in \mathbb{C}$.

Based on the definition of the category $\text{Rep}(S_\nu)$ defined by Deligne, P. Etingof defined in [7] the category $\text{Rep}(H_c(\nu))$:

Definition 5.0.2 (Category $\text{Rep}(H_c(\nu))$). $\text{Rep}(H_c(\nu))$ is the category whose objects are triples (M, x, y) , where M is an ind-object of Deligne's category $\text{Rep}(S_\nu)$, and

$$x : \mathfrak{h}_0^* \otimes M \longrightarrow M$$

$$y : \mathfrak{h}_0 \otimes M \longrightarrow M$$

are morphisms in $\text{Rep}(S_\nu)$ satisfying the conditions:

(1) The morphism

$$x \circ (\text{Id} \otimes x) \circ ((\text{Id} - \sigma_{\mathfrak{h}_0^*, \mathfrak{h}_0^*}) \otimes \text{Id}) : \mathfrak{h}_0^* \otimes \mathfrak{h}_0^* \otimes M \longrightarrow M$$

is 0.

(2) The morphism

$$y \circ (\text{Id} \otimes y) \circ ((\text{Id} - \sigma_{\mathfrak{h}_0, \mathfrak{h}_0}) \otimes \text{Id}) : \mathfrak{h}_0 \otimes \mathfrak{h}_0 \otimes M \longrightarrow M$$

is 0.

(3) The morphism

$$y \circ (\text{Id} \otimes x) - x \circ (\text{Id} \otimes y) \circ (\sigma_{\mathfrak{h}_0, \mathfrak{h}_0^*} \otimes \text{Id}) : \mathfrak{h}_0 \otimes \mathfrak{h}_0^* \otimes M \longrightarrow M$$

equals

$$\delta_{\mathfrak{h}_0} \otimes \text{Id} - \frac{c}{2}(\delta_{\mathfrak{h}_0} \otimes \text{Id}) \circ (\Omega^3 - \Omega^{13} - \Omega^{23} + \Omega^{123})$$

Here

- σ is the symmetry isomorphism in $\text{Rep}(S_\nu)$ (i.e. $\sigma_{U, U'} : U \otimes U' \longrightarrow U' \otimes U$ is the canonical isomorphism in $\text{Rep}(S_\nu)$, where U, U' are objects of $\text{Rep}(S_\nu)$),
- $\delta_{\mathfrak{h}_0} : \mathfrak{h}_0 \otimes \mathfrak{h}_0^* \longrightarrow \mathbb{C}$ is the evaluation map in category $\text{Rep}(S_\nu)$,

- Ω is the generalized Jucys-Murphy element, and the notation Ω^{i_1, \dots, i_k} signifies the operator which acts as Ω on the factors i_1, \dots, i_k of the tensor product, and as Id on the rest.

Morphisms in $\text{Rep}(H_c(\nu))$ are defined to be morphisms of M in $\text{Rep}(S_\nu)$ compatible with maps x, y .

Remark 5.0.3. As in the classical Cherednik algebra, we can replace \mathfrak{h}_0 with \mathfrak{h} in the above definition, and get an equivalent definition of the category $\text{Rep}(H_c(\nu))$.

6. CATEGORY $O_{c,\nu}$

6.1. Definition of $O_{c,\nu}$. Consider the algebra objects $S\mathfrak{h}_0^*, S\mathfrak{h}_0$ in $\text{ind} - \text{Rep}(S_\nu)$. Given maps x, y as above, we get maps

$$\bar{x}_M : S\mathfrak{h}_0^* \otimes M \longrightarrow M$$

$$\bar{y}_M : S\mathfrak{h}_0 \otimes M \longrightarrow M$$

So we can speak of M as an $S\mathfrak{h}_0^*$ -module and an $S\mathfrak{h}_0$ -module in $\text{ind} - \text{Rep}(S_\nu)$.

Definition 6.1.1. The category $O_{c,\nu}$ is defined as the full subcategory of $\text{Rep}(H_c(\nu))$ whose objects are (M, x, y) such that

- M is finitely generated over $S\mathfrak{h}_0^*$ in $\text{Rep}(S_\nu)$ (i.e. M is a quotient of a “finitely generated free $S\mathfrak{h}_0^*$ module” $S\mathfrak{h}_0^* \otimes V$, where V is an object of $\text{Rep}(S_\nu)$).
- M is locally nilpotent over \mathfrak{h}_0 , in the following sense:
for any $\text{Rep}(S_\nu)$ -subobject $T \subset M$, there exists a non-negative integer m (depending on T) such that the map $S^m \mathfrak{h}_0 \otimes T \longrightarrow M$ (this is the restriction to T of the map $S^m \mathfrak{h}_0 \otimes M \rightarrow M$) equals 0.

Definition 6.1.2. Let M be an object in category $O_{c,\nu}$. Define $\mathbf{h} \in \text{End}(\text{Forget}_{O_{c,\nu} \rightarrow \text{Rep}(S_\nu)})$ as $\mathbf{h} = \frac{1}{2}(\psi_1 + \psi_2)$, where $\psi_1, \psi_2 : M \longrightarrow M$,

$$\psi_1 = x \circ (\text{Id} \otimes y) \circ (\text{coev}_{\mathfrak{h}^*} \otimes \text{Id}_M) : M \longrightarrow \mathfrak{h}^* \otimes \mathfrak{h} \otimes M \longrightarrow M$$

$$\psi_2 = y \circ (\text{Id} \otimes x) \circ (\text{coev}_{\mathfrak{h}} \otimes \text{Id}_M) : M \longrightarrow \mathfrak{h} \otimes \mathfrak{h}^* \otimes M \longrightarrow M$$

Similarly to the integer case, we have:

Lemma 6.1.3.

$$y_M \circ (\text{Id}_M \otimes \mathbf{h}) - \mathbf{h} \circ y_M = -y_M : \mathfrak{h}_0 \otimes M \longrightarrow M$$

$$x_M \circ (\text{Id}_M \otimes \mathbf{h}) - \mathbf{h} \circ x_M = x_M : \mathfrak{h}_0^* \otimes M \longrightarrow M$$

From now on, we will consider the generic case, i.e. when $\nu \notin \mathbb{Z}_+$ and $\text{Rep}(S_\nu)$ is semisimple, unless it is stated otherwise. Note that most results can be reformulated to “interpolate” the original result in case $\nu \in \mathbb{Z}_+$.

Definition 6.1.4 (Simple singular subobject). Let $U \in O_{c,\nu}, V \in \text{Rep}(S_\nu)$ so that $V \subset U$ and the map y_U restricted to $\mathfrak{h}_0 \otimes V$ is 0. Then V is called “singular in U ”. If, furthermore, V is a simple object in $\text{Rep}(S_\nu)$, then we say that V is a simple singular $\text{Rep}(S_\nu)$ -subobject of U .

Remark 6.1.5. Note that by definition of $O_{c,\nu}$, each object contains a simple singular $\text{Rep}(S_\nu)$ -subobject.

Definition 6.1.6 (Order of a subobject). Let $U \in O_{c,\nu}, V \in \text{Rep}(S_\nu)$ so that $V \subset U$ and V is simple (in $\text{Rep}(S_\nu)$) and let m be the minimal positive integer such that the map \bar{y}_U restricted to $S^m \mathfrak{h}_0 \otimes V$ is 0. Then V is called “of order m in U ”.

Remark 6.1.7. A singular subobject has order 1.

6.2. Verma objects in category $O_{c,\nu}$. One can define Verma objects in $O_{c,\nu}$ category as follows:

Consider the category $Rep(B_\nu)$ of pairs (M, y_M) , where M is an ind-object of $Rep(S_\nu)$,

$$y : \mathfrak{h}_0 \otimes M \rightarrow M$$

is a morphism in $Rep(S_\nu)$ satisfying the condition: the morphism

$$y \circ (\text{Id} \otimes y) \circ ((\text{Id} - \sigma_{\mathfrak{h}_0, \mathfrak{h}_0}) \otimes \text{Id}) : \mathfrak{h}_0 \otimes \mathfrak{h}_0 \otimes M \longrightarrow M$$

is 0.

This is the analogue of the category of representations of the “Borel” subalgebra $B_n = S\mathfrak{h}_0 \rtimes \mathbb{C}[S_n]$.

Then we have the restriction functor $Res : Rep(H_c(\nu)) \longrightarrow Rep(B_\nu)$, and it has a left adjoint which is the induction functor $Ind : Rep(B_\nu) \longrightarrow Rep(H_c(\nu))$. The induction functor takes a pair (M, y_M) to a triple $(S\mathfrak{h}_0^* \otimes M, y_{S\mathfrak{h}_0^* \otimes M}, x_{S\mathfrak{h}_0^* \otimes M})$, where the map $x_{S\mathfrak{h}_0^* \otimes M} : \mathfrak{h}_0^* \otimes S\mathfrak{h}_0^* \otimes M \rightarrow S\mathfrak{h}_0^* \otimes M$ is the multiplication map of $\mathfrak{h}_0^* \otimes S\mathfrak{h}_0^* \rightarrow S\mathfrak{h}_0^*$ tensored with Id_M , and $y_{S\mathfrak{h}_0^* \otimes M} : \mathfrak{h}_0 \otimes S\mathfrak{h}_0^* \otimes M \rightarrow S\mathfrak{h}_0^* \otimes M$ is a map satisfying conditions (1,3) of Definition 5.0.2, so that $y_{S\mathfrak{h}_0^* \otimes M} |_{\mathfrak{h}_0 \otimes 1 \otimes M} = y_M$.

Definition 6.2.1 (Verma Object). Consider a simple object X_τ in $Rep(S_\nu)$, and let the map $y_{X_\tau} : \mathfrak{h}_0 \otimes X_\tau \rightarrow X_\tau$ be zero. This makes (X_τ, y_{X_τ}) an object of $Rep(B_\nu)$.

Define the Verma object of lowest weight X_τ as $Ind(X_\tau, y_{X_\tau})$ (so it is an ind-object of $Rep(S_\nu)$). It will be denoted by $M_{c,\nu}(\tau)$ (or, if c, ν are fixed, $M(\tau)$ for short).

Remark 6.2.2. It is easy to see that $M_{c,\nu}(\tau) = S\mathfrak{h}_0^* \otimes X_\tau$, where X_τ is a simple object of $Rep(S_\nu)$.

The map $\bar{x}_{M_{c,\nu}(\tau)} : S\mathfrak{h}_0^* \otimes S\mathfrak{h}_0^* \otimes X_\tau \longrightarrow S\mathfrak{h}_0^* \otimes X_\tau$ is the multiplication map of $S\mathfrak{h}_0^*$ tensored with Id_{X_τ} .

The map $\bar{y}_{M_{c,\nu}(\tau)} : S\mathfrak{h}_0 \otimes S\mathfrak{h}_0^* \otimes X_\tau \longrightarrow S\mathfrak{h}_0^* \otimes X_\tau$ is the map for which condition (3) from Definition 5.0.2 holds, and thus is a deformation of the differentiation map $S\mathfrak{h}_0 \otimes S\mathfrak{h}_0^* \longrightarrow S\mathfrak{h}_0^*$ tensored with Id_{X_τ} .

Thus $M(\tau)$ is an object of the category $O_{c,\nu}$.

Notation 6.2.3. To avoid confusion, the Verma module in $O(H_c(n))$ whose lowest weight corresponds to the Young diagram $\tilde{\tau}(n)$ is denoted by $M_{c,n}(\tilde{\tau}(n))$, while the Verma object in $O_{c,\nu=n}$ corresponding to the Young diagram τ is denoted by $M_{c,\nu=n}(\tau)$.

Definition 6.2.4. Let

$$h_{c,\nu}(\tau) := \dim(\mathfrak{h}_0)/2 - c\Omega|_{X_\tau} = \frac{\nu-1}{2} - c \cdot \left(\frac{(\nu-|\tau|)(\nu-|\tau|-1)}{2} - |\tau| + ct(\tau) \right)$$

(here Ω is the generalized Jucys-Murphy element).

Example 6.2.5. • $ct(\tau^s) = \frac{s(s-1)}{2}$, so $h_{c,\nu}(\tau^s) = \frac{\nu-1}{2} - c \cdot \left(\frac{\nu^2-\nu}{2} - s\nu + s^2 - s \right)$.
• $ct(\pi^n) = -\frac{n(n-1)}{2}$, so $h_{c,\nu}(\pi^n) = \frac{\nu-1}{2} - c \cdot \left(\frac{\nu^2-\nu}{2} - n\nu \right)$.

Proposition 6.2.6. For $M = M_{c,\nu}(\tau)$, $\mathbf{h} : M_{c,\nu}(\tau) \longrightarrow M_{c,\nu}(\tau)$ is an endomorphism of $M_{c,\nu}(\tau) = \bigoplus_{m \geq 0} S^m \mathfrak{h}_0^* \otimes X_\tau$ as an ind-object of $Rep(S_\nu)$ such that $\mathbf{h} = \bigoplus_{m \geq 0} \phi_m$, where $\phi_m : S^m \mathfrak{h}_0^* \otimes X_\tau \longrightarrow S^m \mathfrak{h}_0^* \otimes X_\tau$, $\phi_m = (h_{c,\nu}(\tau) + m) \text{Id}$.

Proof. Follows immediately from Property 6.1.3 of the element \mathbf{h} . □

This proposition means that $M(\tau)$ (as a $\text{Rep}(S_\nu)$ ind-object) has a grading by eigenvalues of \mathbf{h} which is the natural \mathbb{Z}_+ -grading on $M(\tau)$ shifted by $h_{c,\nu}(\tau)$.

Now let U be a subquotient of $M(\tau)$. Then U automatically inherits a grading from the grading of $M(\tau)$ by eigenvalues of \mathbf{h} , and the definition of \mathbf{h} implies that maps between subquotients of Verma objects preserve this grading.

For simplicity, we will use the natural \mathbb{Z}_+ -grading on $M(\tau)$ when we refer to the degree in which a $\text{Rep}(S_\nu)$ -object lies in U .

Definition 6.2.7. We say that U is reducible in degree m if m is the smallest positive integer such that there exists a simple singular $\text{Rep}(S_\nu)$ -subobject in U which lies in degree m of U .

Remark 6.2.8. We will see that the definition of reducibility in degree m is a consistent one, i.e. if U contains a simple singular $\text{Rep}(S_\nu)$ -subobject in degree m of U , then there is a subobject of U (in category $O_{c,\nu}$) which lies in degrees m and higher.

Proposition 6.2.9. *Let $U \in O_{c,\nu}$. Let X_μ be a simple singular $\text{Rep}(S_\nu)$ -subobject in U . Then there exists a morphism $M_{c,\nu}(\mu) \rightarrow U$ in $O_{c,\nu}$, inducing Id_{X_μ} on X_μ when regarded as a morphism of $\text{Rep}(S_\nu)$ ind-objects.*

Proof. Recall the definition of a Verma object. X_μ being a simple singular $\text{Rep}(S_\nu)$ -subobject in U means that $\text{Hom}_{\text{Rep}(B_\nu)}(X_\mu, \text{Res}(U)) \neq 0$, where X_μ is considered as a $\text{Rep}(B_\nu)$ -object with $y_{X_\mu} = 0$, and there exists a non-zero map in $\text{Hom}_{\text{Rep}(B_\nu)}(X_\mu, \text{Res}(U))$ whose image is the chosen copy of X_μ in U . Then by definition of a Verma object, we have: $\text{Hom}_{\text{Rep}(B_\nu)}(X_\mu, \text{Res}(U)) = \text{Hom}_{O_{c,\nu}}(M_{c,\nu}(\mu), U) \neq 0$ and there exists a non-zero map in $\text{Hom}_{O_{c,\nu}}(M_{c,\nu}(\mu), U)$ inducing Id_{X_μ} on X_μ when regarded as a morphism of $\text{Rep}(S_\nu)$ ind-objects. \square

This justifies Remark 6.2.8.

Remark 6.2.10. Note that if $\nu \notin \mathbb{Z}_+$, then we would have $\dim\{\phi \in \text{Hom}_{B_\nu}(X_\mu, \text{Res}(U)), \text{Im}(\phi) \text{ is the chosen copy of } X_\mu \text{ in } U\} = 1$, and so $\dim\{\phi \in \text{Hom}_{O_{c,\nu}}(M_{c,\nu}(\mu), U), \phi(X_\mu) \text{ is the chosen copy of } X_\mu \text{ in } U\} = 1$.

Proposition 6.2.11.

- Each Verma object $M_{c,\nu}(\tau)$ has a maximal proper $O_{c,\nu}$ -subobject J , i.e. it has a unique simple quotient $L_{c,\nu}(\tau)$ in $O_{c,\nu}$.
- The simple objects of $O_{c,\nu}$ category are exactly $L_{c,\nu}(\tau)$.

Proof.

- Let J be the sum of all the proper $O_{c,\nu}$ -subobjects of $M_{c,\nu}(\tau)$. For any proper $O_{c,\nu}$ -subobject N of $M_{c,\nu}(\tau)$, N inherits the grading from $M_{c,\nu}(\tau)$ (as a $\text{Rep}(S_\nu)$ ind-object), with \mathbf{h} acting on grade m of N by $(h_{c,\nu}(\tau) + m) \text{Id}$. Since N is a proper subobject, we have $m > 0$. So J can be presented as a direct sum of $\text{Rep}(S_\nu)$ -objects on which the restriction of \mathbf{h} acts by scalars strictly larger than $h_{c,\nu}(\tau)$. This proves that J is a proper $O_{c,\nu}$ -subobject of $M_{c,\nu}(\tau)$.
- Let L be a simple object of $O_{c,\nu}$. Then L has a simple singular $\text{Rep}(S_\nu)$ -subobject X_τ . By Proposition 6.2.9, there exists a non-zero map $M(\tau) \rightarrow L$, with induced map on X_τ being Id_{X_τ} . But L is simple, so $M(\tau) \rightarrow L$ is surjective.

 \square

Remark 6.2.12. Let $M_{c,\nu}(\tau)$ be a Verma object in $O_{c,\nu}$. By the propositions above, to check its reducibility, we only need to check whether there are any non-zero morphisms from other Verma objects to $M_{c,\nu}(\tau)$.

We finish this section with a general observation.

Observation 6.2.13. Let $V \in \text{Rep}(S_\nu)$ be a subobject of a Verma object $M_{c,\nu}(\tau)$, considered as an ind-object of $\text{Rep}(S_\nu)$. $M_{c,\nu}(\tau)$ is defined for any $c, \nu \in \mathbb{C}$, and its decomposition into a sum of indecomposable objects of $\text{Rep}(S_\nu)$ doesn't depend on the values of c, ν .

We would like to describe the set of values c, ν for which V has order less than or equal m in $M_{c,\nu}(\tau)$, i.e. for which the morphism $\bar{y}_{M_{c,\nu}(\tau)}|_{S^m \mathfrak{h}_0 \otimes V}: S^m \mathfrak{h}_0 \otimes V \rightarrow M_{c,\nu}(\tau)$ is 0.

Recall that the category $\text{Rep}(S_\nu)$ is semisimple for generic (non-integer) ν , and so we can write $\mathfrak{h}_0 \otimes V, M_{c,\nu}(\tau)$ as direct sums of simple objects in $\text{Rep}(S_\nu)$ (the summands do not depend on ν, c , as mentioned above), and thus get a natural basis of the \mathbb{C} -vector space $\text{Hom}(S^m \mathfrak{h}_0 \otimes V, M_{c,\nu}(\tau))$ (since Schur's lemma holds in $\text{Rep}(S_\nu)$). Again, the basis doesn't depend on c, ν .

The map $\bar{y}_{M_{c,\nu}(\tau)}|_{S^m \mathfrak{h}_0 \otimes V}$, when written in this basis, has coefficients which depend polynomially on c, ν .

This means that the set of values c, ν for which V has order less than or equal m in $M_{c,\nu}(\tau)$ is a Zariski closed subset of \mathbb{C}^2 .

7. MORPHISMS BETWEEN VERMA OBJECTS: GENERAL REMARKS AND NECESSARY CONDITIONS

7.1. Necessary conditions. The following formula will serve as one of the main tools of this paper.

Proposition 7.1.1. *Let μ be a partition and $m > 0$ an integer, and assume there is a non-zero morphism $M_{c,\nu}(\mu) \rightarrow M_{c,\nu}(\tau)$ such that X_μ (image of the lowest weight of $M(\mu)$) sits in degree m of $M_{c,\nu}(\tau)$. Then*

$$c \left(\frac{|\tau|^2 - |\mu|^2 - (|\tau| - |\mu|)}{2} + ct(\tau) - ct(\mu) \right) = m + (|\tau| - |\mu|)c\nu$$

Proof. Considering the endomorphisms \mathbf{h} of objects $M_{c,\nu}(\mu), M_{c,\nu}(\tau)$, we get:

If there exists a non-trivial morphism $M_{c,\nu}(\mu) \rightarrow M_{c,\nu}(\tau)$, with the image of the lowest weight X_μ of $M_{c,\nu}(\mu)$ lying in degree m of $M_{c,\nu}(\tau)$, then

$$h_{c,\nu}(\tau) + m = h_{c,\nu}(\mu)$$

i.e.

$$\frac{\nu - 1}{2} - c \cdot \left(\frac{(\nu - |\tau|)(\nu - |\tau| - 1)}{2} - |\tau| + ct(\tau) \right) + m = \frac{\nu - 1}{2} - c \cdot \left(\frac{(\nu - |\mu|)(\nu - |\mu| - 1)}{2} - |\mu| + ct(\mu) \right)$$

which can be rewritten as

$$c \cdot \left(\frac{|\tau|^2 - |\mu|^2 - (|\tau| - |\mu|)}{2} - (|\tau| - |\mu|)\nu + ct(\tau) - ct(\mu) \right) = m$$

□

Remark 7.1.2. Note that if $c = 0$, Equation (7.1.1) implies that $m = 0$, $\mu = \tau$ and the morphism $M(\mu) \rightarrow M(\tau)$ is the identity map. This means that for $c = 0$, all the Verma objects are simple. The category $\mathcal{O}(H_0(\nu))$ is a continuation of the categories of modules over $\mathbb{C}S_n \ltimes A_n$, which are \mathcal{O} -coherent D -modules and whose Fourier transform has support $\{0\}$. Here A_n is the n -th Weyl algebra (the algebra of differential operators on \mathbb{C}^n).

From now on, we will assume that $c \neq 0$ and denote:

Notation 7.1.3. $c' := \frac{1}{c}$.

Remark 7.1.4. In this notation, Equation (7.1.1) can be rewritten as

$$\frac{|\tau|^2 - |\mu|^2 - (|\tau| - |\mu|)}{2} + ct(\tau) - ct(\mu) = mc' + (|\tau| - |\mu|)\nu$$

Notation 7.1.5. Let τ, μ be Young diagrams, and m be a positive integer. Denote by $\mathcal{L}_{\tau, \mu, m}$ the set of points (c', ν) in \mathbb{C}^2 satisfying the Equation (7.1.4).

The above proposition shows (with the notations as in Notation 1.1.1): $\mathcal{B}_{\tau, \mu} \subset \biguplus_{m \in \mathbb{Z}_{>0}} \mathcal{L}_{\tau, \mu, m}$.

Remark 7.1.6. Note that Proposition 7.1.1 implies the following statement:

Fix c, ν , and consider the lowest weight X_μ of the Verma object $M_{c, \nu}(\mu)$. Then for all non-trivial maps $M_{c, \nu}(\mu) \rightarrow M_{c, \nu}(\tau)$ the degree m of $M_{c, \nu}(\tau)$ in which X_μ sits is the same, since it is given by Equation (7.1.4).

For example, we have the following lemma (for its statement and proof when ν is an integer, see [10, Lemma 3.5]):

Lemma 7.1.7. *Let $X_\mu \subset \mathfrak{h}_0 \otimes X_\tau$ in $\text{Rep}(S_\nu)$. We can regard X_μ as sitting in degree 1 of the $O_{c, \nu}$ -object $M_{c, \nu}(\mu)$. Then there is a morphism $M_{c, \nu}(\mu) \rightarrow M_{c, \nu}(\tau)$ in $O_{c, \nu}$, inducing Id_{X_μ} on X_μ , if and only if*

$$\frac{|\tau|^2 - |\mu|^2 - (|\tau| - |\mu|)}{2} + ct(\tau) - ct(\mu) = c' + (|\tau| - |\mu|)\nu$$

Proof. Consider the morphism $Y : \mathfrak{h}_0^* \otimes X_\tau \rightarrow \mathfrak{h}_0^* \otimes X_\tau$ which is equivalent to $y|_{\mathfrak{h}_0^* \otimes X_\tau} : \mathfrak{h}_0 \otimes \mathfrak{h}_0^* \otimes X_\tau \rightarrow X_\tau$ under the isomorphism $\text{End}(\mathfrak{h}_0^* \otimes X_\tau) \cong \text{Hom}(\mathfrak{h}_0 \otimes \mathfrak{h}_0^* \otimes X_\tau, X_\tau)$.

In $O(H_c(n))$, we have:

$$Y = 1 - c \sum_{s \in S} 1 \otimes s + c \sum_{s \in S} s \otimes s$$

so in our case, we would have:

$$Y = 1 - c\Omega^2 - c\Omega^{1,2} = 1 + h_{c, \nu}(\tau) - \frac{\nu - 1}{2} - c\Omega^{1,2}$$

(Ω as in Definition 5.0.2). Thus Y acts on a simple object $X_\mu \subset \mathfrak{h}_0 \otimes X_\tau$ by

$$1 + h_{c, \nu}(\tau) - \frac{\nu - 1}{2} - \left(h_{c, \nu}(\mu) - \frac{\nu - 1}{2} \right) = 1 + h_{c, \nu}(\tau) - h_{c, \nu}(\mu)$$

This proves that X_μ is a simple singular $\text{Rep}(S_\nu)$ -subobject of $M(\tau)$ iff $1 + h_{c, \nu}(\tau) - h_{c, \nu}(\mu) = 0$, which is equivalent to

$$\frac{|\tau|^2 - |\mu|^2 - (|\tau| - |\mu|)}{2} + ct(\tau) - ct(\mu) = c' + (|\tau| - |\mu|)\nu$$

□

An additional condition for X_μ to sit in degree m of $M_{c, \nu}(\tau)$ arises from Pieri's rule (cf. Proposition 4.2.1).

In terms of Notation 1.1.1, Lemma 7.1.7 means that $\mathcal{B}_{\tau, \mu, 1} = \mathcal{L}_{\tau, \mu, 1}$ whenever μ is obtained from τ as in Pieri's rule, i.e. by adding/moving/deleting a cell, or if $\mu = \tau$.

7.2. Remarks on Equation (7.1.4). For a Young diagram μ , denote: $f(\mu) := \frac{|\mu|^2 - |\mu|}{2} + ct(\mu)$. Then Equation (7.1.4) can be rewritten as

$$(1) \quad f(\tau) - f(\mu) = c'm + (|\tau| - |\mu|)\nu$$

Lemma 7.2.1. *Let μ be a Young diagram. Then $f(\mu)$ is a non-negative integer less or equal to $|\mu|^2 - |\mu|$, and it is equal to 0 if μ is a column diagram, and to $|\mu|^2 - |\mu|$ if μ is a row diagram.*

Proof. This is equivalent to saying that $|ct(\mu)| \leq \frac{|\mu|^2 - |\mu|}{2}$. The latter can be proved by induction on the number of columns of μ :

Base: Assume the number of columns of μ is 0, i.e. $\mu = \emptyset$. Then $|ct(\mu)| = 0 = \frac{|\mu|^2 - |\mu|}{2}$. Also, if μ is a column diagram, then $ct(\mu) = -\frac{|\mu|^2 - |\mu|}{2}$.

Step: Denote by k the number of columns of μ ($k > 1$), by μ' the diagram μ without the last column, and by l the number of boxes in the last column of μ . By induction assumption, $|ct(\mu')| \leq \frac{|\mu'|^2 - |\mu'|}{2}$. Next, $ct(\mu) = ct(\mu') + (k-1) \cdot l - l(l-1)/2$, and so $\frac{|\mu|^2 - |\mu|}{2} = \frac{|\mu'|^2 - |\mu'|}{2} + |\mu'|l + \frac{l^2 - l}{2} \geq |ct(\mu')| + \frac{l^2 - l}{2} + |\mu'|l \geq |ct(\mu') + (k-1) \cdot l - l(l-1)/2| = |ct(\mu)|$ (for the last inequality, note that by definition, $|\mu'| \geq (k-1)$, with equality if and only if μ is a row diagram). For a row diagram μ (with k cells), there is an equality $ct(\mu) = \frac{|\mu|^2 - |\mu|}{2}$, and in general, $|ct(\mu)| > \frac{|\mu|^2 - |\mu|}{2}$ for μ having $k > 1$ columns and not a row diagram. □

For the right hand side of Equation (1), note that we have, by Pieri's rule (Proposition 4.2.1): $m \geq ||\mu| - |\tau||$.

8. BLOCKS OF CATEGORY $O(H_c(n))$ (CLASSICAL CASE)

8.1. KZ functor and connection to the representations of the Hecke algebra.

A powerful tool in studying the O category for Cherednik algebra is the KZ functor (see e.g. [14][2.8.2], [9][Section 6], [2][Section 3.3]). The KZ functor is a functor $O(H_c(n)) \rightarrow \text{Rep}(\mathcal{H}_q(n))$, $\text{Rep}(\mathcal{H}_q(n))$ being the category of representations of the Hecke algebra of type A, where $q = \exp(2\pi i c)$.

It turns out that this functor is essentially surjective on objects, surjective on Homs and exact (see [12]). This functor induces an equivalence of categories $\overline{KZ} : O/O^{tor} \rightarrow \text{Rep}(\mathcal{H}_q(n))$, where O^{tor} is a full subcategory of the O category whose objects are modules which, when considered as $\mathbb{C}[\mathfrak{h}_0]$ -modules, have Krull dimension less than $n-1$ (see [9][Section 6.3]). The non-zero objects of O/O^{tor} would thus be $H_c(n)$ -modules which are supported on the whole \mathbb{C}^n .

Moreover, by the result mentioned in [2][Proposition 3.4], the KZ functor is faithful on the full subcategory of Verma modules in $O(H_c(n))$.

We will use the fact that the KZ functor takes $M(\lambda)$ to the dual of the Specht module $S_{\tilde{\lambda}}$ if $c \geq 0$, and to S_{λ} (Specht module) if $c < 0$. For $c < 0$, the module $L(\lambda)$ goes to D_{λ} (which is either the unique simple quotient of the Specht module S_{λ} , if it exists, or 0).

8.2. Representations of Hecke algebra of type A. We will use the following facts about the representations of Hecke algebra of type A (see [18], [2][Section 3]):

Some definitions:

Notation 8.2.1. Denote $e := \text{ord}_{\mathbb{C}}(q)$ (i.e. e is the order of q if q is a root of unity and ∞ otherwise), and denote $s := n - e$.

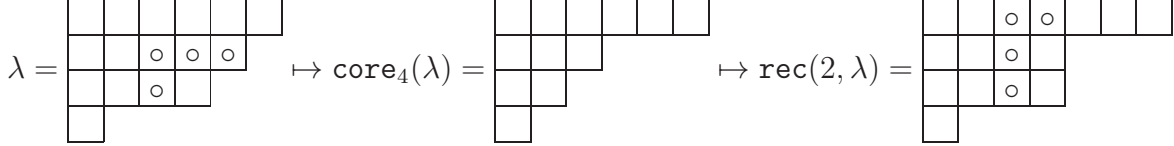
Proof. Basically, $\text{rec}(l, \lambda)$ is obtained by "inserting" a $\text{hook}(l, (n-s))$ to λ in a certain not completely trivial way. To insert a hook $\text{hook}(l, (n-s))$ into a diagram λ , one should put its vertex in the position (i, j) such that the following equations are satisfied:

$$\lambda_i + 1 \leq i - 1 + n - s - l \leq \lambda_{i-1}$$

$$\check{\lambda}_j + 1 \leq j + l \leq \check{\lambda}_{j-1}$$

This shows that there exists exactly one way to insert the hook into λ . \square

Example 8.2.9. Let $\lambda = (6, 5, 4, 1)$. Adding the 4-hook $(2, 1, 1)$ to its 4-core ($\text{core}_4(\lambda) = (6, 3, 2, 1)$), we obtain the Young diagram $\text{rec}(2, \lambda) = (7, 4, 4, 1)$:



Theorem 8.2.10 (cf. [18]). *Let τ, μ be partitions of n .*

- *All the simple modules of $\mathcal{H}_q(n)$ are D_λ , where λ is e -regular (i.e. the multiplicities of all parts of λ are smaller than e). These are exactly the partitions for which $D_\lambda \neq 0$.*
- *The $\mathcal{H}_q(n)$ -modules S_τ^\sim and S_τ have the same composition factors.*
- *Two modules S_τ, S_μ belong to the same block iff τ, μ have the same $(n-s)$ -core.*
- *If τ has no $(n-s)$ -hook, then S_τ (and hence S_τ^\sim) is irreducible.*
- *Let $\tau = \text{rec}(l, \beta)$, $0 \leq l \leq (n-s-2)$ and $\mu \neq \tau$. Then there is a non-trivial morphism $S(\mu) \rightarrow S(\tau)$ iff $\mu = \text{rec}(l+1, \beta)$.*
- *If $\tau = \text{rec}(l, \beta)$, $\mu = \text{rec}(l+1, \beta)$, then the composition factors of S_τ are D_τ, D_μ , with multiplicity 1 (only one of them if the other is 0).*

Remark 8.2.11. Note that only the last two cases use the assumption that $n > 2s$.

8.3. Blocks of category $O(H_c(n))$. We now give the relevant results for the category $O(H_c(n))$:

Let n be a non-negative integer.

8.3.1. Equivalences. First of all, we have a useful theorem proved by Rouquier and expanded by Losev (see [20, Theorem 5.12], [16]):

Theorem 8.3.1 (Rouquier). *Let $n > 1, r, a > 0, b \neq 0$ be integers, $\text{gcd}(r, a) = 1$. Then the categories $O(H_{c_1=b/a}(n)), O(H_{c_2=(br)/a}(n))$ are equivalent if $c_1 \notin \{\frac{2k+1}{2}, k \in \mathbb{Z}\}$, with a correspondence:*

morphism $M_{c_1, n}(\mu) \rightarrow M_{c_1, n}(\tau)$ ($|\tau| = |\mu| = n$) such that μ sits in degree m of $M_{c_1, n}(\tau)$ corresponds to
morphism $M_{c_1, n}(\mu) \rightarrow M_{c_1, n}(\tau)$ ($|\tau| = |\mu| = n$) such that μ sits in degree rm of $M_{c_1, n}(\tau)$.

Next, we have the following simple equivalence of categories: (see [21][3.1.4]):

Observation 8.3.2. The rational Cherednik algebras $H_{-c}(n), H_c(n)$ are isomorphic:

$$H_c(n) \longrightarrow H_{(-c)}(n), x \longmapsto x, y \longmapsto y, \sigma \in S_n \longmapsto (\text{sign}(\sigma) \cdot \sigma) \in \mathbb{C}[S_n]$$

This means that the categories of representations of these algebras are equivalent, with equivalence given by

$$O(H_c(n)) \longrightarrow O(H_{(-c)}(n)), V \longmapsto \text{sign} \otimes V,$$

where $sign$ is the sign representation of S_n . Note that $sign \otimes \mu \cong \check{\mu}$ for representation μ of S_n .

We also have the following statement (see [12][Section 6.2]):

Proposition 8.3.3. *If $c \notin \frac{1}{2} + \mathbb{Z}$, then*

$$Hom_{H_c(n)}(M_{c,n}(\mu), M_{c,n}(\tau)) \cong Hom_{H_{(-c)}(n)}(M_{(-c),n}(\tau), M_{(-c),n}(\mu))$$

8.3.2. *Results on $O(H_c(n))$ obtained from the theory of representations of the Hecke algebra $\mathcal{H}_q(n)$.* The correspondence between the lowest weight representations of $H_c(n)$ and the finite dimensional representations of the Hecke algebra $\mathcal{H}_q(n)$ gives us the following theory (see [8]):

First of all, we have (see [9][6.13]):

Proposition 8.3.4. *For $c \leq 0$, $KZ(M_{c,n}(\tau)) = S(\tau)$, $KZ(L_{c,n}(\tau)) = D(\tau)$, and thus $KZ(L_{c,n}(\tau)) \neq 0$ iff τ is $(n-s)$ -regular. For $c > 0$, $KZ(L_{c,n}(\tau)) \neq 0$ iff τ is $(n-s)$ -restricted.*

We next have the following theorem:

Theorem 8.3.5 (Dipper, James; cf. [6]). *If c' satisfies one of the following conditions:*

- $c' \notin \mathbb{Q}$,
- $c' \in \mathbb{Q}$, and $c' = \frac{d}{a}$, $\mathbf{gcd}(a, d) = 1$, $d > n$,

then the category $O(H_c(n))$ is semisimple (in particular, all Verma modules are simple).

Proof. This is a direct consequence of Theorem 8.2.2, the standard property of Verma modules (Proposition 6.2.9) and the fact that the KZ functor is faithful on the full subcategory of Verma modules in $O(H_c(n))$. \square

This means that we remain with only one interesting case: $c' = \frac{d}{a}$, $\mathbf{gcd}(a, d) = 1$, $a > 0$, $0 \leq d \leq n$. For $d \leq n/2$, the theory is more complicated, but for $d > n/2$, it is rather simple and explained below.

Fix integer $s \geq 0$, and from now on, consider $n > 2s$, $c' = n - s$, and a Young diagram τ with $|\tau| = n$.

Corollary 8.3.6. *$M(\tau)$ is simple if and only if τ has no $(n-s)$ -hook, or its $(n-s)$ -hook is a vertical strip.*

Proof. This is a consequence of Theorem 8.2.10 and of the standard property of Verma modules (Proposition 6.2.9). \square

Furthermore, we have (see [2][Sections 3.4, 3.5]):

Theorem 8.3.7. *Let $\tau = \mathbf{rec}(l, \beta)$, $0 \leq l \leq (n-s-2)$ and $\mu \neq \tau$. Then there is a non-trivial morphism $\psi_l : M(\mu) \rightarrow M(\tau)$ iff $\mu = \mathbf{rec}(l+1, \beta)$.*

In that case, we have:

- $\dim \text{Hom}(M(\mu), M(\tau)) = 1$.
- *There is a short exact sequence*

$$0 \longrightarrow L(\mu) \longrightarrow M(\tau) \longrightarrow L(\tau) \longrightarrow 0.$$

Remark 8.3.8. In [2][Section 3], this is proved for the case $s = 0$, i.e. $e = n$. The only thing which we need to understand to generalize this claim is the following:

The method used to prove that there's such a short exact sequence uses the fact that for $c = n/r$, $r > 0$, $\mathbf{gcd}(n, r) = 1$, $L(\lambda)$ is thin (i.e. doesn't have full support, and goes

to 0 under KZ) iff $\lambda = \emptyset$. But for $c = r/(n-s)$, we have other thin modules (actually, those are exactly $L(\lambda)$ such that λ is not $(n-s)$ -restricted), which can appear.

Suppose we fix a core with s boxes, call it C , and assume $n \gg s$. Then we have a chain of Young diagrams of n boxes with $(n-s)$ -core C : $\lambda^{(0)} \leftarrow \lambda^{(1)} \leftarrow \lambda^{(2)} \leftarrow \dots$, where $\lambda_{(0)}$ has the largest content. $\lambda_{(0)}$ is obtained by simply adding a strip of $(n-s)$ boxes to the right of C , so it is not $(n-s)$ -restricted. But on the contrary, all $\lambda^{(i)}$ for $i > 0$ are in fact $(n-s)$ -restricted. In this case an argument similar to the one in [2] would work (as diagrams with different cores cannot appear, since they lie in different blocks).

Example 8.3.9. The Verma module $M_{c,n}(\tau^n)$ (whose lowest weight is the trivial representation $\tau^n = \tilde{\emptyset}(n)$ of S_n) is reducible if and only if the following condition on $c' = \frac{1}{c}, n$ holds:

$$c' = \frac{n-s}{r}, s \geq 0, r \geq 1$$

In this case, $M_{c,n}(\tau^n)$ contains a lowest weight $H_c(n)$ -submodule, whose lowest weight μ lies in degree $r(s+1)$ of $M_{c,n}(\tau^n)$. The Young diagram of μ is a two row diagram, the lower row containing $s+1$ boxes.

Theorem 8.3.7 gives the following corollary:

Corollary 8.3.10. *Let $n > 3, s \geq 0, r \geq 1$ be integers, $n > 2s, \gcd(r, n-s) = 1$. Then the blocks of the category $O(H_{c=\frac{r}{n-s}}(n))$ correspond to $(n-s)$ -cores, and we have two types of blocks: blocks containing only one Verma module (up to isomorphism), corresponding to a diagram which doesn't have an $(n-s)$ -hook, and blocks which correspond to a Young diagrams of size s ; in such a block Block_β ($|\beta| = s$) lie the Verma modules whose $(n-s)$ -core is β , and these Verma modules form a long exact sequence:*

$$\begin{aligned} 0 \longrightarrow M(\text{rec}((n-s-1), \beta)) &\xrightarrow{\psi_{(n-s-2)}} M(\text{rec}((n-s-2), \beta)) \xrightarrow{\psi_{(n-s-3)}} \dots \\ \dots &\xrightarrow{\psi_1} M(\text{rec}(1, \beta)) \xrightarrow{\psi_0} M(\text{rec}(0, \beta)) \longrightarrow L(\text{rec}(0, \beta)) \longrightarrow 0 \end{aligned}$$

(Note that $\text{rec}((n-s-1), \beta)$ is the diagram β with a vertical $(n-s)$ -hook added, and $\text{rec}(0, \beta)$ is the diagram β with a horizontal $(n-s)$ -hook added).

Remark 8.3.11. For $s \geq n/2$, the situation is more complicated, but one can still say the following (see [8]): Let $c' = n-s$.

- One can define the $(n-s)$ -core of τ as the diagram obtained from τ by removing all its $(n-s)$ -hooks.
- If τ has no $(n-s)$ -hooks, then $M(\tau)$ is a simple module, and the block where it lies is semisimple.
- If $M(\mu), M(\tau)$ belong to the same block, then they have the same $(n-s)$ -cores.

9. CONSTRUCTIONS FOR $O_{c,\nu}$

We now define constructions analogous to those described in Section 8 for the category $O_{c,\nu}$, where we consider generic values of ν . We use the idea described in Section 4 of treating a simple object X_τ of $\text{Rep}(S_\nu)$ as a Young diagram obtained by adding a very long top row to τ (“the top row having length $(\nu - |\tau|)$ ”). This idea is only meant to provide some intuition, but it is true that for $n \in \mathbb{Z}, n \gg 0$, the image of X_τ under the functor $\text{Rep}(S_{\nu=n}) \rightarrow \text{Rep}(S_n)$ is isomorphic to $\tilde{\tau}(n)$, which is exactly a Young diagram obtained by adding a top row of length $(n - |\tau|)$ to τ . So instead of adding and removing $(n-s)$ -hooks as we did in Section 8, we would be adding and removing “ $(\nu-s)$ -hooks”

from diagrams with “very long top row of size $(\nu - |\tau|)$ ”. Below is a formal description of the relevant constructions.

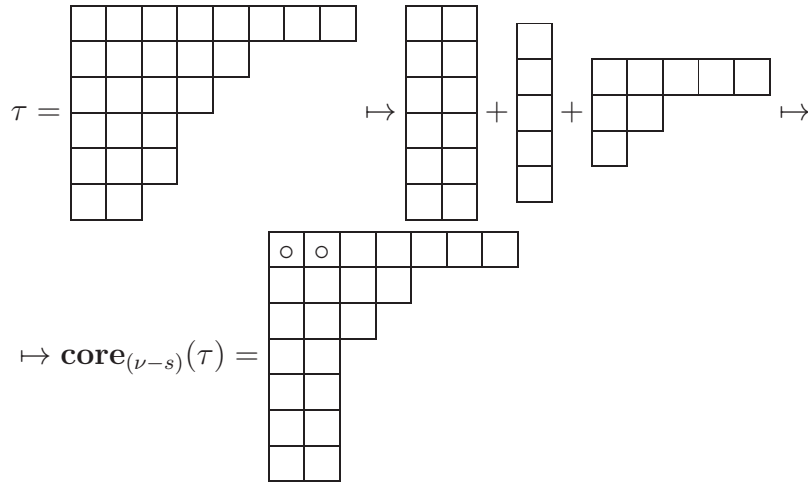
Let τ be a Young diagram, and $s \geq 0$ be an integer.

Definition 9.0.12.

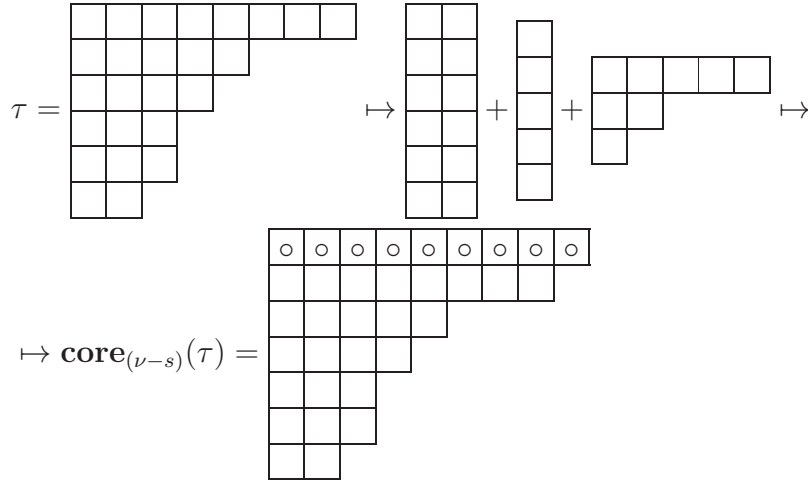
- Define $C_\tau = \{|\tau| - 1 + j - \check{\tau}_j \mid j \geq 1\}$.
- For $s \in C_\tau$, $s = |\tau| - 1 + j_s - \check{\tau}_{j_s}$ for some $j_s \geq 1$, define the $(\nu - s)$ -core of τ as $\mathbf{core}_{(\nu-s)}(\tau)_{\check{j}} = \check{\tau}_j + 1$ if $1 \leq j < j_s$, and $\mathbf{core}_{(\nu-s)}(\tau)_{\check{j}} = \check{\tau}_{j+1}$ if $j \geq j_s$.

That is, $\mathbf{core}_{(\nu-s)}(\tau)$ is the Young diagram obtained by taking out column j_s of τ , moving columns $1, \dots, j_s - 1$ down and adding a row of length $j_s - 1$ on top of them, and moving left the columns $j_s + 1, \dots$.

Example 9.0.13. Let $\tau = (8, 5, 4, 3, 3, 2)$. Then $|\tau| = 25$, $C_\tau = \{19, 20, 22, 25, 27, 29, 30, 31\} \cup \mathbb{Z}_{\geq 33}$. Let $s = 22$. Then $s \in C_\tau$, $s = |\tau| - 1 + 3 - \check{\tau}_3$. So $j_s = 3$, and $\mathbf{core}_{(\nu-s)}(\tau) = (7, 4, 3, 2, 2, 2)$.



Let $s = 34$. Then $s \in C_\tau$, $s = |\tau| - 1 + 10 - \check{\tau}_{10}$. So $j_s = 10$, $\mathbf{core}_{(\nu-s)}(\tau) = (9, 8, 5, 4, 3, 3, 2)$.



Remark 9.0.14. Deligne defined $\mathbf{core}_{(\nu-s)}(\lambda)$, which he denoted by $\{\lambda\}_n^+$, with $n := s$, in [5, 7.5]. He also shows, in [5, Lemma 7.3] that $\mathbb{Z}_+ \setminus C_\tau$ is the set of zeroes of polynomial P_τ which occurs in the formula

$$\dim X_\tau = \dim(\tau) \frac{\prod_{k \in \mathbb{Z}_+ \setminus C_\tau} (\nu - k)}{|\tau|!}$$

where $\dim(\tau)$ is the dimension of the irreducible representation of $S_{|\tau|}$ corresponding to Young diagram τ .

The process of “reconstruction” (corresponding to inserting a hook into a Young diagram) is defined as follows:

Construction 9.0.15. Given any Young diagram η and an integer $l \geq 0$, we define $\tau := \mathbf{rec}(l, \eta)$ as the Young diagram obtained through the following steps:

- Find the index $k \geq 1$ such that $\eta_{k-1} \geq l + 1 > \eta_k$ (here $\eta_0 := \infty$).
- Define $\tau := \mathbf{rec}(l, \eta)$, $\tau_j = \eta_j - 1$ for $j < k$, $\tau_k = l$, $\tau_j = \eta_{j-1}$ for $j > k$.

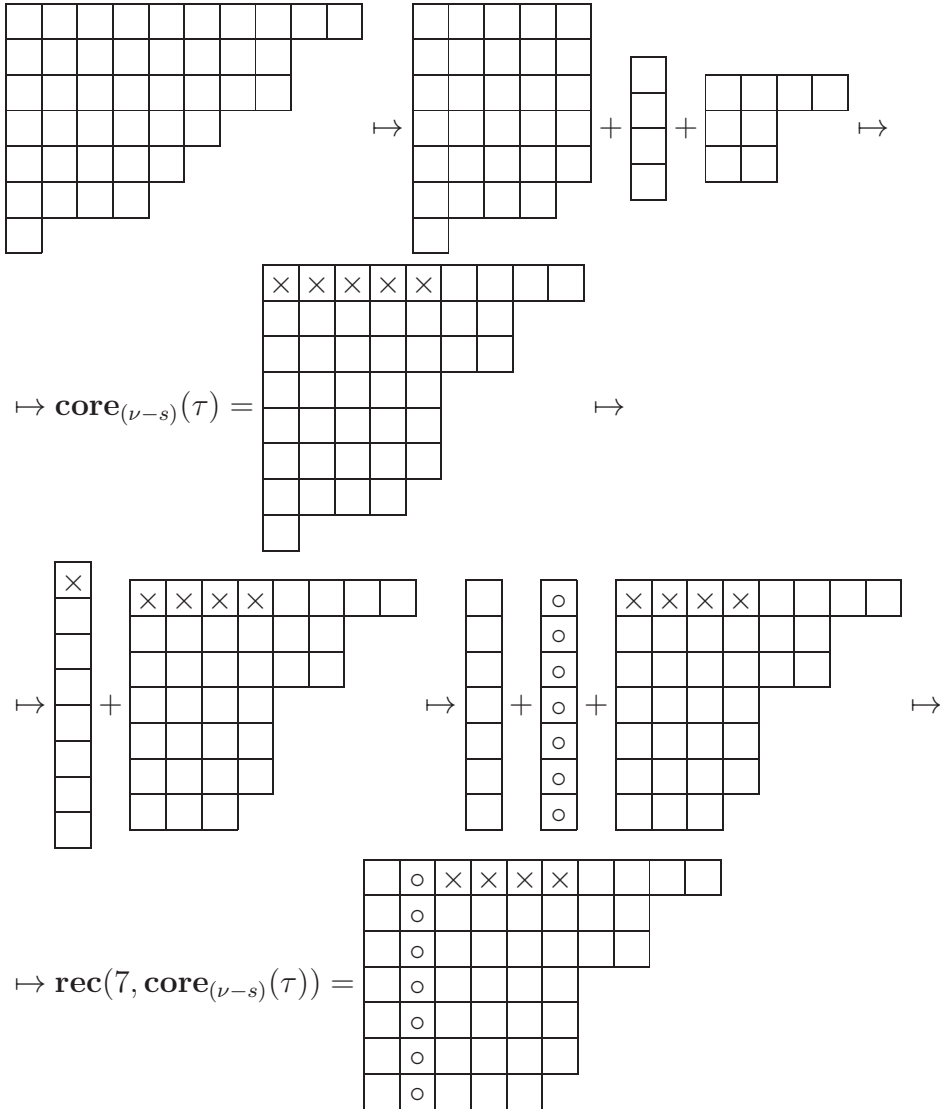
That is, we divide η into two parts: part 1 consisting of columns $1, \dots, k-1$ and part 2 consisting of columns $k, k+1, \dots$. Then we delete the top row of part 1, add a k -th column of length l , and add part 2 as columns $k+1, k+2, \dots$.

Example 9.0.16. • For $l > \check{\eta}_1$, $\mathbf{rec}(l, \eta)$ is the Young diagram obtained from η by adding a column of length l to η (this will become the first column).

- For $l = 0$, $\mathbf{rec}(l, \eta)$ is the Young diagram obtained from η by removing its top row.

Example 9.0.17. $\mathbf{rec}(\tau_{j_s}, \mathbf{core}_{(\nu-s)}(\tau)) = \tau$.

Example 9.0.18. Let $\tau = (10, 8, 8, 6, 5, 4, 1)$, $s = 42$ (so $j_s = 6$). Then $\mathbf{rec}(7, \mathbf{core}_{(\nu-s)}(\tau)) = (10, 8, 8, 6, 6, 6, 5)$.



Example 9.0.19. Consider $\tau = \emptyset$. Given $s \geq 0$, the $(\nu - s)$ -core of \emptyset is a row of length s . Then $\mathbf{rec}(l, \mathbf{core}_{(\nu-s)}(\emptyset))$ is a hook with arm length s and leg length $l - 1$.

Lemma 9.0.20. Let τ be a Young diagram, $l > 0$ an integer, $s := |\tau| - 1 + j_s - \tilde{\tau}_{j_s}$ for some $j_s \geq 1$, $\mu = \mathbf{rec}(\tilde{\tau}_{j_s} + l, \mathbf{core}_{(\nu-s)}(\tau))$. Then $|\mu| - |\tau| > 0$, and $s = \frac{f(\mu) - f(\tau)}{|\mu| - |\tau|}$ (f is defined in Subsection 7.2).

Proof. We first describe μ in terms of τ . By Constructions 9.0.12, 9.0.15 described above, μ is obtained from τ by taking out column j_s of τ , inserting a column of length $\tilde{\tau}_{j_s} + l$ after column $k_{s,l} - 1$ for some $k_{s,l} \leq j_s$ ($k_{s,l}$ is uniquely determined by Construction 9.0.15), and adding a cell to each of the columns $k_{s,l} + 1, \dots, j_s$ of the newly constructed diagram. Thus $\check{\mu}_j = \check{\tau}_j$ for $j < k_{s,l}$ and $j > j_s$, $\check{\mu}_{k_{s,l}} = \check{\tau}_{j_s} + l$, and $\check{\mu}_j = \check{\tau}_j + 1$ for $j = k_{s,l} + 1, \dots, j_s$. This means that we have:

$$|\mu| - |\tau| = j_s - k_{s,l} + l,$$

Note that since $k_{s,l} \leq j_s$, $l > 0$, we have: $|\mu| - |\tau| > 0$.

So

$$\begin{aligned} ct(\mu) - ct(\tau) &= - \left(j_s \tilde{\tau}_{j_s} - \frac{\tilde{\tau}_{j_s}(\tilde{\tau}_{j_s} + 1)}{2} \right) + \left(\frac{j_s(j_s - 1)}{2} - \frac{k_{s,l}(k_{s,l} - 1)}{2} \right) + \dots \\ &\dots + \left(k_{s,l}(\tilde{\tau}_{j_s} + l) - \frac{(\tilde{\tau}_{j_s} + l)(\tilde{\tau}_{j_s} + l + 1)}{2} \right) = \dots \\ &\dots = -\tilde{\tau}_{j_s}(j_s - k_{s,l} + l) + \frac{j_s(j_s - 1)}{2} - \frac{k_{s,l}(k_{s,l} - 1)}{2} - \frac{l(l + 1)}{2} + k_{s,l}l = \dots \\ &\dots = -\tilde{\tau}_{j_s}(j_s - k_{s,l} + l) - \frac{j_s - k_{s,l} + l}{2} + \frac{j_s^2 - (k_{s,l} - l)^2}{2} = \dots \\ &\dots = - \left(\tilde{\tau}_{j_s} + \frac{1}{2} \right) (j_s - k_{s,l} + l) + \frac{(j_s + k_{s,l} - l)(j_s - k_{s,l} + l)}{2} \end{aligned}$$

And thus

$$\frac{ct(\mu) - ct(\tau)}{|\mu| - |\tau|} = -\tilde{\tau}_{j_s} - \frac{1}{2} + \frac{j_s + k_{s,l} - l}{2}$$

Recall that we also have, by definition:

$$\frac{f(\mu) - f(\tau)}{|\mu| - |\tau|} = \frac{|\mu| + |\tau| - 1}{2} + \frac{ct(\mu) - ct(\tau)}{|\mu| - |\tau|}$$

And so

$$\begin{aligned} \frac{f(\mu) - f(\tau)}{|\mu| - |\tau|} &= \frac{|\mu| + |\tau| - 1}{2} - \tilde{\tau}_{j_s} - \frac{1}{2} + \frac{j_s + k_{s,l} - l}{2} = \dots \\ &\dots = \frac{2|\tau| + j_s - k_{s,l} + l - 1}{2} - \tilde{\tau}_{j_s} - \frac{1}{2} + \frac{j_s + k_{s,l} - l}{2} = |\tau| - 1 + j_s - \tilde{\tau}_{j_s} = s \end{aligned}$$

□

These constructions are compatible with the constructions described in Section 8 in the following sense:

Proposition 9.0.21. Let $n \gg 0$. Put $\lambda^{(l)} := \mathbf{rec}(l, \mathbf{core}_{(\nu-s)}(\tau))$. Then $\widetilde{\lambda^{(l)}}(n) = \mathbf{rec}(l, \mathbf{core}_{(n-s)}(\tilde{\tau}(n)))$.

Proof. One can easily see that the procedure for constructing $\mathbf{rec}(l, \mathbf{core}_{(\nu-s)}(\tau))$ coincides with the procedure for constructing $\mathbf{rec}(l, \mathbf{core}_{(\nu-s)}(\tilde{\tau}(n)))$ and then removing the top row. \square

10. BLOCKS IN CATEGORY $O_{c,\nu}$

10.1. Relation between $O_{c,\nu=n}$ and $O(H_c(n))$.

Remark 10.1.1. Note that since $\text{Rep}(S_n)$ is $\text{Rep}(S_{\nu=n})$ quotiented by an ideal (of objects whose dimension is 0), given a non-trivial morphism $M_{c,\nu=n}(\mu) \rightarrow M_{c,\nu=n}(\tau)$, the induced morphism $M_{c,n}(\tilde{\mu}(n)) \rightarrow M_{c,n}(\tilde{\tau}(n))$ might be 0, but if $n > 2|\mu|$, then the latter morphism will not be 0 either (since the object X_μ will not lie in the above ideal).

In general, given a morphism $f : U_1 \rightarrow U_2$ in $O_{c,\nu}$, and a simple object X_μ in U_1 , we have: if for $\nu = n$, the restriction of f to X_μ is not 0, and $n > 2|\mu|$, then the morphism f induces a non-zero morphism in $O(H_c(n))$.

Remark 10.1.2. Similarly, if there is a simple $\text{Rep}(S_\nu)$ -subobject X_μ of $U \in O_{c,\nu=n}$, and X_μ has order $\leq m$, then in $O(H_c(n))$, the corresponding subobject has order $\leq m$ as well (but might become 0, unless $n > 2|\mu|$).

Conversely, if the S_n -submodule $\tilde{\mu}(n)$ of $U \in O(H_c(n))$ has order $\leq m$ (i.e. the vectors inside $\tilde{\mu}(n)$ are all killed by some polynomial in y_i 's of degree no more than m), and $n > 2|\mu|$, then in $O_{c,\nu=n}$, the $\text{Rep}(S_\nu)$ -subobject X_μ of U has order $\leq m$ as well.

10.2. Morphisms between two Verma objects. We now give some necessary and some sufficient conditions for the existence of a non-trivial morphism between Verma objects. Fix Young diagrams τ, μ .

Proposition 10.2.1. *Let $r \in \mathbb{Z} \setminus \{0\}$, $s \in C_\tau$ (i.e. $s = |\tau| - 1 + j_s - \check{\tau}_{j_s}$ for some $j_s \geq 1$).*

Let $\mu = \mathbf{rec}(\tilde{\tau}_{j_s} + \text{sign}(r), \mathbf{core}_{(\nu-s)}(\tau))$.

If $c' = \frac{\nu-s}{r}$, then there exists a non-trivial morphism $M_{c,\nu}(\mu) \rightarrow M_{c,\nu}(\tau)$.

Proof. Assume $(c', \nu), \mu$ are as above.

Let $n \gg 0$ be an integer, such that $\mathbf{gcd}(n-s, r) = 1$ (there are infinitely many such positive integers). By Corollary 8.3.10 and by Proposition 9.0.21, we have a non-trivial morphism $M_{c,n}(\tilde{\mu}(n)) \rightarrow M_{c,n}(\tilde{\tau}(n))$. Moreover, $s = \frac{f(\mu)-f(\tau)}{|\mu|-|\tau|}$ (f defined in Subsection 7.2) by Lemma 9.0.20, and the image of $\tilde{\mu}(n)$ sits in degree $m := |r|(|\mu|-|\tau|)$ of $M_{c,n}(\tilde{\tau}(n))$ (this is a direct consequence of Equation (1)).

Then by Subsection 10.1, there is a non-trivial morphism $M_{c,\nu=n}(\mu) \rightarrow M_{c,\nu=n}(\tau)$ with the image of X_μ sitting in degree $m = |r|(|\mu|-|\tau|)$ of $M_{c,\nu=n}(\tau)$, and $s = \frac{f(\mu)-f(\tau)}{|\mu|-|\tau|}$.

Following the ideas of Observation 6.2.13, we get the required statement. \square

Remark 10.2.2. In the proof of this proposition, we established that $s = \frac{f(\mu)-f(\tau)}{|\mu|-|\tau|}$, so by Equation (1), the image of X_μ sits in degree m of $M_{c,\nu}(\tau)$ for all (c', ν) such that $c' = \frac{\nu-s}{r}$. In general, for each pair (c', ν) , Equation (1) allows us to compute the unique degree m in which the image of X_μ could sit in $M_{c,\nu}(\tau)$.

We continue with Young diagrams τ, μ fixed.

Consider the line $\mathcal{L}_{\tau,\mu,m} \subset \mathbb{C}^2$ (defined in 7.1.5). We would like to check whether $\mathcal{L}_{\tau,\mu,m}$ satisfies the following condition:

Condition 10.2.3. For all ν there exist non-trivial morphisms $\Theta_\nu : M_{c,\nu}(\mu) \rightarrow M_{c,\nu}(\tau)$ such that $(c', \nu) \in \mathcal{L}_{\tau,\mu,m}$ and $\Theta_\nu(X_\mu)$ sits in degree m of $M_{c,\nu}(\tau)$.

Assume this is indeed the case.

Then by Subsection 10.1, for integer $n \gg 0$ there must exist non-trivial morphisms $\tilde{\Theta}_n : M_{c,n}(\tilde{\mu}(n)) \rightarrow M_{c,n}(\tilde{\tau}(n))$ (corresponding to $\Theta_{\nu=n}$) with the image of $\tilde{\mu}(n)$ sitting in degree m of $M_{c,n}(\tilde{\tau}(n))$.

Recall that for $n \in \mathbb{Z}_+$, $(c', n) \in \mathcal{L}_{\tau,\mu,m}$ is equivalent to $c' = \frac{(|\mu| - |\tau|)n - (f(\mu) - f(\tau))}{m}$. Denote: $a := |\mu| - |\tau|$, $b := f(\mu) - f(\tau)$, $d_n := \mathbf{gcd}(an - b, m)$. Then $c' = \frac{an - b}{m} = \frac{(an - b)/d_n}{m/d_n}$, where $(an - b)/d_n, m/d_n \in \mathbb{Z}, m/d_n > 0, \mathbf{gcd}((an - b)/d_n, m/d_n) = 1$.

By Theorem 8.3.1, if $(an - b)/d_n \neq 2$, we can pass to the case $(c' = (an - b)/d_n, n)$, with the image of $\tilde{\mu}(n)$ sitting in degree d_n of $M_{c,n}(\tilde{\tau}(n))$.

Note that if $a \neq 0$, then, since $0 < d_n \leq m$, we have $(an - b)/d_n \neq 2$ for $n \gg 0$.

Now we have the following cases:

Case $a > 0$.

Lemma 10.2.4. *For $n \gg 0$, there are no non-trivial morphisms $\tilde{\Theta}_n$ unless $a \mid b, m$, i.e. $d_n = a$.*

Proof. We assume these morphisms are not trivial. Then we have, by Theorem 8.3.5, $(an - b)/d_n < n$ for $n \gg 0$, i.e. $a \leq d_n$. Also, putting $d := \mathbf{gcd}(a, b)$, $a' := a/d, b' := b/d$, we have: $d_n = \mathbf{gcd}(d(a'n - b'), m) \geq a = da' \geq 1$ for $n \gg 0$. This implies, for $n \gg 0$: either $d_n \mid d$ (possible only if $d_n = d = a$, and then we have $a \mid b, m$), or $\mathbf{gcd}(a'n - b', m) \neq 1$.

So we only need to check that there is no $n_0 \in \mathbb{Z}$ such that $\forall n > n_0, \mathbf{gcd}(a'n - b', m) \neq 1$. Indeed, $b', \mathbf{gcd}(a', m)$ are relatively prime (since a', b' are), and so $\mathbf{gcd}(a'n - b', m) = \mathbf{gcd}(\frac{a'}{\mathbf{gcd}(a', m)}n - b', \frac{m}{\mathbf{gcd}(a', m)})$. Now

$$\mathbf{gcd}(\frac{a'}{\mathbf{gcd}(a', m)}, b') = \mathbf{gcd}(\frac{a'}{\mathbf{gcd}(a', m)}, \frac{m}{d'}) = 1$$

so

$$\mathbf{gcd}(a'n - b', m) = \mathbf{gcd}(\frac{a'}{\mathbf{gcd}(a', m)}n - b', \frac{m}{\mathbf{gcd}(a', m)}) = 1$$

for infinitely many integer values of n . \square

Thus $d_n = a$, and $(an - b)/d_n \neq 2$ for $n \gg 0$. So to understand what happens for $(c', n) \in \mathcal{L}_{\tau,\mu,m}$, it is enough to check what happens the case $(c' = n - (b/a), n)$ (recall: $b/a = \frac{f(\mu) - f(\tau)}{|\mu| - |\tau|} \in \mathbb{Z}$). Still assuming the existence of non-trivial morphisms $\tilde{\Theta}_n$, Proposition 9.0.21 and Corollary 8.3.10 gives:

For $n \gg 0$ and $c' = n - b/a$, put $s := b/a$, and we have

$$\mu = \mathbf{rec}(\tilde{\tau}_{j_s} + 1, \mathbf{core}_s(\tau)) \text{ for } j_s \geq 1, \text{ such that } s = |\tau| - 1 + j_s - \tilde{\tau}_{j_s}$$

Notice that a priori, j_s could depend on n , but the equality $s = |\tau| - 1 + j_s - \tilde{\tau}_{j_s}$ defines j_s uniquely for each s , so since s doesn't depend on n , neither does j_s .

Conclusion 10.2.5. For $|\mu| > |\tau|$, $\mathcal{L}_{\tau,\mu,m}$ satisfies Condition 10.2.3 if and only if the following hold:

- (1) $s := \frac{f(\mu) - f(\tau)}{|\mu| - |\tau|} \in C_\tau \subset \mathbb{Z}$,
- (2) $\mu = \mathbf{rec}(\tilde{\tau}_{j_s} + 1, \mathbf{core}_{(\nu-s)}(\tau))$,
- (3) $(|\mu| - |\tau|) \mid m$.

In that case, points $(c', \nu) \in \mathcal{L}_{\tau,\mu,m}$ satisfy the condition: $c' = \frac{\nu-s}{r}$ for $r = \frac{m}{|\mu| - |\tau|}$.

Case $a < 0$. By Observation 8.3.2, the existence of a non-trivial morphism $\tilde{\Theta}_n$ for $n \gg 0$ is equivalent to the existence of a non-trivial morphism $\tilde{\Upsilon}_n : M_{-c',n}(\tilde{\tau}(n)) \rightarrow M_{-c',n}(\tilde{\mu}(n))$. The necessary and sufficient conditions for the existence of a non-trivial morphism $\tilde{\Upsilon}_n$ for all $n \gg 0$ are given in Conclusion 10.2.5, with μ and τ switching roles.

Conclusion 10.2.6. For $|\mu| < |\tau|$, $\mathcal{L}_{\tau,\mu,m}$ satisfies Condition 10.2.3 if and only if the following hold:

- (1) $s := \frac{f(\mu)-f(\tau)}{|\mu|-|\tau|} \in C_\mu \subset \mathbb{Z}$,
- (2) $\tau = \mathbf{rec}(\tilde{\mu}_{j_s} + 1, \mathbf{core}_{(\nu-s)}(\mu))$,
- (3) $(|\tau| - |\mu|) \mid m$.

In that case, points $(c', \nu) \in \mathcal{L}_{\tau,\mu,m}$ satisfy the condition: $c' = -\frac{\nu-s}{r}$ for $r = \frac{m}{|\tau|-|\mu|}$.

Case $a = 0$. In this case, we are considering lines $c' = \frac{f(\tau)-f(\mu)}{m} = \frac{ct(\tau)-ct(\mu)}{m}$, i.e. c' is constant. Put $d := \mathbf{gcd}(ct(\tau) - ct(\mu), m)$ (previously denoted d_n ; now doesn't depend on n). By definition, $0 < d \leq m$. If $d \neq \frac{ct(\tau)-ct(\mu)}{2}$, Theorem 8.3.1 allows us to pass to the case $c' = (ct(\tau) - ct(\mu))/d \in \mathbb{Z}$ (i.e. we will now consider the line $\mathcal{L}_{\tau,\mu,m/d} = \{(c' = (ct(\tau) - ct(\mu))/d, \nu) \mid \nu \in \mathbb{C}\}$).

From now on, we will assume $d \neq \frac{ct(\tau)-ct(\mu)}{2}$.

It would be difficult to determine whether morphisms $\tilde{\Theta}_n$ exist for

$(c', n) \in \mathcal{L}_{\tau,\mu,m/d}, n \in \mathbb{Z}, n \gg 0$, but we can rule out some cases using the tools we have so far. First of all, if there exists an integer $n > 2|\mu| = 2|\tau|$ such that $c' = (ct(\tau) - ct(\mu))/d > n$ (i.e. if $(ct(\tau) - ct(\mu))/d > 2|\tau| + 1$), then by Theorem 8.3.5, there are no non-trivial morphisms $\tilde{\Theta}_n$, and we are done. Theorem 8.3.5 also tells us that the same argument applies to the case $c' = (ct(\tau) - ct(\mu))/d < -(2|\tau| + 1)$, so we ruled out the possibilities $|c'| = |(ct(\tau) - ct(\mu))/d| > 2|\tau| + 1$.

We will show that something can be said if $|\tau| < |(ct(\tau) - ct(\mu))/d| \leq 2|\tau| + 1$.

For this, we should consider the sign of $ct(\tau) - ct(\mu)$.

Case 1. Let $ct(\tau) - ct(\mu) > 0, |\tau| < (ct(\tau) - ct(\mu))/d \leq 2|\tau| + 1$. Let $n := 2|\tau| + 1$.

Assume we have a non-trivial morphism $\tilde{\Theta}_n$. Then, putting $s := n - (ct(\tau) - ct(\mu))/d$, we have: $n > 2s$, so by Corollary 8.3.10 and Proposition 9.0.21,

$$\mu = \mathbf{rec}(\tilde{\tau}_{j_s} + 1, \mathbf{core}_s(\tau)) \text{ for } j_s \geq 1, \text{ such that } s = |\tau| - 1 + j_s - \tilde{\tau}_{j_s}$$

But then we would get, by Lemma 9.0.20: $|\mu| > |\tau|$. Contradiction.

Case 2. Let $ct(\tau) - ct(\mu) < 0, |\tau| < (ct(\tau) - ct(\mu))/d \leq 2|\tau| + 1$. Let $n := 2|\tau| + 1$. By Proposition 8.3.2, we can pass to the previous case (Case 10.2) by switching the roles of μ and τ , so we conclude there are no non-trivial morphisms $\tilde{\Theta}_n$ in that case.

Conclusion 10.2.7. For $|\tau| = |\mu|$, if $\frac{ct(\tau)-ct(\mu)}{\mathbf{gcd}(ct(\tau)-ct(\mu),m)} > |\tau|$ then $\mathcal{L}_{\tau,\mu,m}$ does not satisfy Condition 10.2.3.

The results of this subsection prove the following theorem:

Theorem 10.2.8. For two Young diagrams μ, τ and an integer $m > 0$, the following are equivalent:

- (1) $|\mu| \neq |\tau|$ and $\mathcal{L}_{\tau,\mu,m} \subset \mathcal{B}_{\mu,\tau}$,
- (2) $\mu = \mathbf{rec}(\tilde{\tau}_{j_s} + \text{sign}(|\mu| - |\tau|), \mathbf{core}_{(\nu-s)}(\tau))$ for some $s \in C_\tau$, and j_s given by $s = |\tau| - 1 + j_s - \tilde{\tau}_{j_s}$.

Remark 10.2.9. We know that for every $c \in \mathbb{C}, n \in \mathbb{Z}_{>0}$,

$$\dim \text{Hom}(M_{c,n}(\tilde{\mu}(n)), M_{c,n}(\tilde{\tau}(n))) \leq 1$$

for any μ, τ .

Similarly to the arguments in this subsection, Subsection 10.1 implies that $\dim \text{Hom}(M_{c,\nu=n}(\mu), M_{c,\nu=n}(\tau)) \leq 1$ for $n \in \mathbb{Z}_+, n \gg 0$, so we have: $\{(c, \nu) | \dim \text{Hom}(M_{c,\nu}(\mu), M_{c,\nu}(\tau)) \geq 1\}$ is a Zariski closed set in \mathbb{C}^2 of dimension at most 1, and the set $\{(c, \nu) | \dim \text{Hom}(M_{c,\nu}(\mu), M_{c,\nu}(\tau)) \geq 2\}$ is a finite set of points in \mathbb{C}^2 .

Remark 10.2.10. The results in this subsection agree with the statement of Lemma 7.1.7 when $m = 1$, but they imply the the statement of Lemma 7.1.7 only if $|\mu| \neq |\tau|$.

10.3. Properties of lines $\mathcal{L}_{\tau,\mu,m}$. We conclude this section with a list of (almost trivial) properties of the lines $\mathcal{L}_{\tau,\mu,m}$:

Lemma 10.3.1. • $\mathcal{L}_{\tau,\mu,m_1} \cap \mathcal{L}_{\tau,\mu,m_2} \neq \emptyset \Leftrightarrow m_1 = m_2$.

- If $(c', \nu) \in \mathcal{L}_{\tau,\mu^1,m_1} \cap \mathcal{L}_{\tau,\mu^2,m_2}$, and the lines $\mathcal{L}_{\tau,\mu^1,m_1}, \mathcal{L}_{\tau,\mu^2,m_2}$ do not coincide, then $c', \nu \in \mathbb{Q}$ (and there is only one point (c', ν) like this).
- Assume the lines $\mathcal{L}_{\tau,\mu^1,m_1}, \mathcal{L}_{\tau,\mu^2,m_2}$ coincide, $|\mu^1| \neq |\tau|$, and $\mathcal{L}_{\tau,\mu^1,m_1} \subset \mathcal{B}_{\mu^1,\tau}$. Then $\mathcal{L}_{\tau,\mu^2,m_2} \subset \mathcal{B}_{\mu^2,\tau}$ iff $\mu^1 = \mu^2$.

Proof. • Follows immediately from the definition of $\mathcal{L}_{\tau,\mu,m}$ (see Notation 7.1.5).

- Follows immediately from the definition of $\mathcal{L}_{\tau,\mu,m}$ by a linear equation with rational coefficients.
- First, $|\mu^1| \neq |\tau|$, and $\mathcal{L}_{\tau,\mu^1,m_1} \subset \mathcal{B}_{\mu^1,\tau}$, so by Theorem 10.2.8, putting

$$s := \frac{f(\mu^1) - f(\tau)}{|\mu^1| - |\tau|}$$

we get

$$\mu^1 = \text{rec}(\tau_{j_s}^{\check{}} + \text{sign}(|\mu^1| - |\tau|), \text{core}_{(\nu-s)}(\tau))$$

with j_s given by $s = |\tau| - 1 + j_s - \tau_{j_s}^{\check{}}$.

Next, since the lines $\mathcal{L}_{\tau,\mu^1,m_1}, \mathcal{L}_{\tau,\mu^2,m_2}$ coincide and $|\mu^1| \neq |\tau|$, we get:

$$[m_1 : (|\mu^1| - |\tau|) : f(\mu^1) - f(\tau)] = [m_2 : (|\mu^2| - |\tau|) : f(\mu^2) - f(\tau)]$$

$$\text{which implies } s = \frac{f(\mu^1) - f(\tau)}{|\mu^1| - |\tau|} = \frac{f(\mu^2) - f(\tau)}{|\mu^2| - |\tau|}$$

In particular, $|\mu^2| \neq |\tau|$ and $\text{sign}(|\mu^1| - |\tau|) = \text{sign}(|\mu^2| - |\tau|)$.

Now assume that $\mathcal{L}_{\tau,\mu^2,m_2} \subset \mathcal{B}_{\mu^2,\tau}$. Since

$$|\mu^2| \neq |\tau|, s = \frac{f(\mu^1) - f(\tau)}{|\mu^1| - |\tau|} = \frac{f(\mu^2) - f(\tau)}{|\mu^2| - |\tau|} \text{ and } \text{sign}(|\mu^1| - |\tau|) = \text{sign}(|\mu^2| - |\tau|)$$

we get, from :

$$\mu^2 = \text{rec}(\tau_{j_s}^{\check{}} + \text{sign}(|\mu^2| - |\tau|), \text{core}_{(\nu-s)}(\tau)) = \mu^1$$

□

From the third part of the above lemma, we immediately get:

Corollary 10.3.2. Let $|\mu| \neq |\tau|$, and $\mathcal{L}_{\tau,\mu,m} \subset \mathcal{B}_{\mu,\tau}$. Then for a generic point $(1/c, \nu) \in \mathcal{L}_{\tau,\mu,m}$, $(1/c, \nu) \notin \mathcal{B}_{\mu',\tau}$ for any $\mu' \neq \mu$.

11. CHARACTERS OF SIMPLE OBJECTS IN $O_{c,\nu}$

In this section, we will assume $\nu \notin \mathbb{Z}, c \neq 0$. We will also identify \mathfrak{h}_0 with \mathfrak{h}_0^* , \mathfrak{h} with \mathfrak{h}^* (since any object in category $Rep(S_\nu)$ is isomorphic to its dual).

11.1. Definitions.

Definition 11.1.1 (Character of an ind-object of $Rep(S_\nu)$). The character of an ind-object V of $Rep(S_\nu)$ is defined to be a formal power series in t :

$$ch_t V := \sum_{\text{Young diagrams } \mu \text{ of arbitrary size}} X_\mu t^{h_{c,\nu}(\mu)} \dim \text{Hom}_{S_\nu}(X_\mu, V)$$

with complex degrees, and coefficients in $K_0(Rep(S_\nu))$, which is the Grothendieck ring of $Rep(S_\nu)$ (for the definition of $h_{c,\nu}$, see Definition 6.2.4).

We will usually write chV instead of $ch_t V$ for short.

11.2. Characters of Verma objects. Recall that $M(\lambda) \cong X_\lambda \otimes S\mathfrak{h}_0$ as ind-objects of $Rep(S_\nu)$. So it is enough to compute the character of $S\mathfrak{h}_0$, and then use the formula

$$chM(\lambda) = chS\mathfrak{h}_0 X_\lambda t^{h_{c,\nu}(\lambda)}$$

We now give a formula for computing the character of the ind-object $S\mathfrak{h}_0$, which comes from the character formula in [9][3.7].

Proposition 11.2.1. *We have the following formula for the character of $S\mathfrak{h}_0$:*

$$chS\mathfrak{h}_0 = \frac{1}{\sum_{n \geq 0} (-1)^n \Lambda^n \mathfrak{h}_0 t^n}$$

Proof. Consider the category of Schur functors $Schur$ of all Schur functors. This category is equivalent to the category $\bigoplus_{n \in \mathbb{Z}} Rep(S_n)$, and thus semisimple.

One can see that $Fun(Schur, Rep(S_\nu)) \cong Rep(S_\nu)$ (this is true for any symmetric tensor category), where $Fun(Schur, Rep(S_\nu))$ are functors between additive linear tensor categories.

Fix the object V in $Schur$ corresponding to the identity functor (which is, of course, a Schur functor as well). In $Schur$, we have a commutative algebra SV .

We then have an exact Koszul-type complex in $Schur \cong \bigoplus_{n \in \mathbb{Z}} Rep(S_n)$:

$$\dots \longrightarrow \Lambda^m V \otimes SV \longrightarrow \Lambda^{m-1} V \otimes SV \longrightarrow \dots \longrightarrow V \otimes SV \longrightarrow SV \longrightarrow \mathbb{C} \longrightarrow 0$$

This gives us an exact complex of ind-objects of $Rep(S_\nu)$:

$$\dots \longrightarrow \Lambda^m \mathfrak{h}_0 \otimes SV \longrightarrow \Lambda^{m-1} \mathfrak{h}_0 \otimes S\mathfrak{h}_0 \longrightarrow \dots \longrightarrow \mathfrak{h}_0 \otimes S\mathfrak{h}_0 \longrightarrow S\mathfrak{h}_0 \longrightarrow \mathbb{C} \longrightarrow 0$$

Now the character formula follows directly from Euler's formula applied to this complex. \square

11.3. The graded space $Hom_{Rep(S_\nu)}(X_\mu, S\mathfrak{h}_0 \otimes X_\tau)$. In this subsection we describe explicitly the decomposition of $S\mathfrak{h} \otimes X_\tau$ and $S\mathfrak{h}_0 \otimes X_\tau$ as $Rep(S_n)$ ind-objects into graded sums of simple objects.

Definition 11.3.1. The character of a graded vector space $V = \bigoplus_{k \in \mathbb{Z}} V_k$ will be defined as $ch_q(V) := \sum_k q^k \dim V_k$.

Lemma 11.3.2. *For $n \gg 0$ (in fact, for $n > 2(|\mu| + |\tau|)$), we have:*

$$ch_q(Hom_{S_n}(\tilde{\mu}(n), \mathbb{C}[x_1, \dots, x_n] \otimes \tilde{\tau}(n))) = (s_{\tilde{\tau}(n)} * s_{\tilde{\mu}(n)})(1, q, q^2, \dots)$$

The latter expression can be rewritten as

$$\begin{aligned} ch_q(Hom_{S_n}(\tilde{\mu}(n), \mathbb{C}[x_1, \dots, x_n] \otimes \tilde{\tau}(n))) &= \sum_{\lambda, |\lambda| + \lambda_1 \leq n} \gamma_{\tilde{\tau}(n), \tilde{\mu}(n)}^{\tilde{\lambda}(n)} s_{\tilde{\lambda}(n)}(1, q, q^2, \dots) = \dots \\ &= \sum_{\lambda} \bar{\gamma}_{\tau, \mu}^{\lambda} s_{\tilde{\lambda}(n)}(1, q, q^2, \dots) \end{aligned}$$

Here

- $s_{\tilde{\lambda}(n)}$ is the Schur symmetric function corresponding to the partition $\tilde{\lambda}(n)$ of n (see [17, Chapter I, Par. 3, p.41]),
- $s_{\tilde{\tau}(n)} * s_{\tilde{\mu}(n)}$ is the internal product of Schur symmetric functions as defined in [17, Chapter I, Par. 7, p.116],
- $\gamma_{\alpha', \alpha''}^{\alpha'''} := \frac{1}{n!} \sum_{w \in S_n} \chi^{\alpha'}(w) \chi^{\alpha''}(w) \chi^{\alpha'''}(w)$ is the Kronecker coefficient of partitions $\alpha', \alpha'', \alpha'''$ of n (here $\chi^{\alpha'}(w)$ is the value at w of the character of the irreducible representation of S_n corresponding to α'),
- $\bar{\gamma}_{\tau, \mu}^{\lambda}$ is the reduced Kronecker coefficient (equals $\gamma_{\tilde{\tau}(n), \tilde{\mu}(n)}^{\tilde{\lambda}(n)}$ for $n \gg 0$). A good reference for standard and reduced Kronecker coefficients is [3].

Proof. We define the map \mathcal{F} (denoted by ch in [17, Chapter I, Par. 7]) from conjugation-invariant functions on S_n to symmetric functions in countably many variables by putting

$$\mathcal{F}(f) := \frac{1}{n!} \sum_{w \in S_n} f(w) \prod_{j \geq 1} \left(\sum_i x_i^{\rho_j(w)} \right)$$

where $\rho(w) = (\rho_1, \rho_2, \dots)$ is the cycle-type of w .

Denoting

$$p_{\rho} := \prod_{j \geq 1} \left(\sum_i x_i^{\rho_j} \right), m(\rho)_i \text{ is the number of parts of size } i \text{ in } \rho, z_{\rho} := \prod_{i \geq 1} i^{m(\rho)_i} m(\rho)_i!$$

we get:

$$\mathcal{F}(f) := \sum_{\rho \text{ is a partition of } n} \frac{1}{z_{\rho}} f(\rho) p_{\rho}$$

(here $f(\rho)$ is the value of f on the conjugacy class of S_n consisting of permutations of cycle-type ρ).

Let V be a representation of S_n . Denote by χ^V the character of V , and by abuse of notation, $\mathcal{F}(V) := \mathcal{F}(\chi^V)$. If $V = \bigoplus_j V_j$ is a \mathbb{Z} -graded representation of S_n , then put $\mathcal{F}_q(V) := \sum_{j \in \mathbb{Z}} \mathcal{F}(\chi^{V_j}) q^j$.

We have: $\mathcal{F}(\chi^{\alpha}) = s_{\alpha}$ for partition α of n , and $\mathcal{F}(\chi^{V'} \chi^{V''}) =: \mathcal{F}(V') * \mathcal{F}(V'')$ ($\chi^{V'} \chi^{V''}$ is the character of the representation $\alpha' \otimes \alpha''$ of S_n).

Denote by g_w the action of $w \in S_n$ on the n -dimensional complex vector space \mathbb{C}^n , g_w given by the permutation matrix corresponding to w .

By MacMahon's Master theorem (see [9, Lemma 3.28]; the proof relies on an argument similar to the one used in 11.2), we have:

$$\begin{aligned} \mathcal{F}_q(\mathbb{C}[x_1, \dots, x_n]) &= \frac{1}{n!} \sum_{w \in S_n} \sum_{k \geq 0} tr(Sym^k(g_w)) q^k p_{\rho(w)} = \frac{1}{n!} \sum_{w \in S_n} \frac{1}{\det(1 - qg_w)} p_{\rho(w)} = \dots \\ &= \frac{1}{n!} \sum_{w \in S_n} \prod_{1 \leq k \leq l(\rho(w))} \frac{1}{(1 - q^{\rho(w)_k})} p_{\rho(w)} = \sum_{\rho \text{ are part. of } n} \frac{1}{z_{\rho}} \prod_{1 \leq k \leq l(\rho)} \frac{1}{(1 - q^{\rho_k})} p_{\rho} \end{aligned}$$

Similarly,

$$\begin{aligned}\mathcal{F}_q(\mathbb{C}[x_1, \dots, x_n] \otimes \tilde{\tau}(n)) &= \sum_{\rho \text{ are part. of } n} \frac{1}{z_\rho} \prod_{1 \leq k \leq l(\rho)} \frac{1}{(1 - q^{\rho_k})} p_\rho \chi^{\tilde{\tau}(n)}(\rho) = \dots \\ \dots &= \sum_{\alpha, \rho \text{ are partitions of } n} \frac{1}{z_\rho} \prod_{1 \leq k \leq l(\rho)} \frac{1}{(1 - q^{\rho_k})} \chi^\alpha(\rho) \chi^{\tilde{\tau}(n)}(\rho) s_\alpha\end{aligned}$$

Thus

$$\begin{aligned}ch_q(Hom_{S_n}(\tilde{\mu}(n), \mathbb{C}[x_1, \dots, x_n] \otimes \tilde{\tau}(n))) &= \text{coefficient of } s_{\tilde{\mu}(n)} \text{ in } \mathcal{F}_q(\mathbb{C}[x_1, \dots, x_n] \otimes \tilde{\tau}(n)) = \dots \\ \dots &= \sum_{\rho \text{ are partitions of } n} \frac{1}{z_\rho} \prod_{1 \leq k \leq l(\rho)} \frac{1}{(1 - q^{\rho_k})} \chi^{\tilde{\mu}(n)}(\rho) \chi^{\tilde{\tau}(n)}(\rho) = (s_{\tilde{\tau}(n)} * s_{\tilde{\mu}(n)})(1, q, q^2, \dots)\end{aligned}$$

The last equality holds by [17, Chapter 6, par. 8., p.363]. \square

Corollary 11.3.3. *Taking \mathfrak{h}_0 to be the reflection representation of S_n , we get:*

$$ch_q(Hom_{S_n}(\tilde{\mu}(n), S\mathfrak{h}_0 \otimes \tilde{\tau}(n))) = (1 - q)(s_{\tilde{\tau}(n)} * s_{\tilde{\mu}(n)})(1, q, q^2, \dots) = (1 - q) \sum_{\lambda} \bar{\gamma}_{\tau, \mu}^\lambda s_{\tilde{\lambda}(n)}(1, q, q^2, \dots)$$

Proof. This follows directly from the fact that $\mathbb{C}[x_1, \dots, x_n] = S\mathfrak{h}$, where $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathbb{C}$ is the permutation representation of S_n , and so $S^m \mathfrak{h} = \bigoplus_{0 \leq j \leq m} S^j \mathfrak{h}_0$. \square

Corollary 11.3.4.

$$ch_q(Hom_{S_\nu}(X_\mu, S\mathfrak{h}_0 \otimes X_\tau)) = (1 - q) \sum_{\lambda \text{ is a partition}} \bar{\gamma}_{\tau, \mu}^\lambda \bar{s}_\lambda(1, q, q^2, \dots)$$

where $\bar{\gamma}_{\tau, \mu}^\lambda$ is the reduced Kronecker coefficient, and

$$\bar{s}_\lambda(1, q, q^2, \dots) := \frac{q^{|\lambda|} s_\lambda(1, q, q^2, \dots)}{\prod_{j \geq 1} (1 - q^j)}$$

Proof. By the structure of Deligne's category described in Section 4, if there exist integers k, N such that for any $n \geq N$, $\dim \text{Hom}_{S_n}(\tilde{\mu}(n), S^m \mathfrak{h}_0) = k$ (here for each n , \mathfrak{h}_0 is the reflection representation of S_n), then $\dim \text{Hom}_{S_\nu}(X_\mu, S^m \mathfrak{h}_0) = k$ in Deligne's category $\text{Rep}(S_\nu)$ as well.

Thus the power series in $ch_q(Hom_{S_\nu}(X_\mu, S\mathfrak{h}_0 \otimes X_\tau))$ is the q -adic limit, as n tends to infinity, of $ch_q(Hom_{S_n}(\tilde{\mu}(n), S\mathfrak{h}_0 \otimes \tilde{\tau}(n)))$; recall that, by definition, a sequence $\{P^{(n)}(q)\}_n$, $P^{(n)}(q) = \sum_{k \geq 0} a_k^{(n)} q^k$ of formal power series in q converges, in the q -adic metric, to the formal power series $P(q)$ if for any $k \geq 0$ there exists $N_k \in \mathbb{Z}_{>0}$ such that for any $n > N_k$, $q^{k+1} \mid (P(q) - P^{(n)}(q))$.

By [17, Chapter I, Par. 5, Example 1],

$$s_\lambda(1, q, q^2, \dots) = \frac{q^{\sum_{i \geq 1} i \lambda_i - |\lambda|}}{\prod_{x \in \lambda} (1 - q^{h(x)})}$$

where $h(x)$ is the size of the hook in λ with vertex x , so

$$s_{\tilde{\lambda}(n)}(1, q, q^2, \dots) = \frac{q^{\sum_i i \lambda_i}}{\prod_{x \in \lambda} (1 - q^{h(x)})} \cdot \prod_{1 \leq j \leq n - |\mu|} \frac{1}{1 - q^{n - |\mu| - j + 1 + \tilde{\mu}_j}}$$

The q -adic limit of $s_{\tilde{\lambda}(n)}(1, q, q^2, \dots)$, as n tends to infinity, is then

$$\bar{s}_{\lambda}(1, q, q^2, \dots) := \frac{q^{\sum_i i \lambda_i}}{\prod_{x \in \lambda} (1 - q^{h(x)})} \cdot \frac{1}{\prod_{j \geq 1} (1 - q^j)} = \frac{q^{|\lambda|} s_{\lambda}(1, q, q^2, \dots)}{\prod_{j \geq 1} (1 - q^j)}$$

This completes the proof of the statement. \square

Example 11.3.5. For $\tau = \emptyset$, we get:

$$\begin{aligned} ch_q(Hom_{S_{\nu}}(X_{\mu}, S\mathfrak{h}_0)) &= (1 - q) \bar{s}_{\mu}(1, q, q^2, \dots) = \frac{q^{|\mu|} s_{\mu}(1, q, q^2, \dots)}{\prod_{j \geq 2} (1 - q^j)} = \dots \\ \dots &= \frac{q^{\sum_{i \geq 1} \mu_i i}}{\prod_{x \in \mu} (1 - q^{h(x)}) \prod_{j \geq 2} (1 - q^j)} \end{aligned}$$

11.4. Characters of simple objects: generic cases.

11.4.1. Fix a Young diagram τ . Recall that for each Young diagram $\mu \neq \tau$ and $m \in \mathbb{Z}_+$, there is a set of points (c', ν) for which $M_{c, \nu}(\mu)$ maps to $M_{c, \nu}(\tau)$ in degree m . These sets are either straight lines or finite sets.

The union of these sets is the reducibility locus of $M_{c, \nu}(\tau)$, denoted by B_{τ} , and outside this reducibility locus, $L_{c, \nu}(\tau) = M_{c, \nu}(\tau)$, and the character of this object is given by the formula

$$chL(\tau) = chM(\tau) = \frac{t^{h_{c, \nu}(\tau)} X_{\tau}}{\sum_{n \geq 0} (-1)^n \Lambda^n \mathfrak{h}_0 t^n}$$

This is the most generic case.

11.4.2. The next most generic case is a generic point on a line $\mathcal{L}_{\tau, \mu, m} \subset \mathcal{B}_{\mu, \tau}$.

Fix a pair s, r such that $s, r \in \mathbb{Z}, r \neq 0, s \geq 0$, and consider a generic point (c', ν) on the line $c' = \frac{\nu - s}{r}$.

Let τ be a Young diagram such that $s \in C_{\tau}$ (this exactly means that $M(\tau)$ is actually reducible at this point (c, ν)), and denote $\mu := \mathbf{rec}(\tilde{\tau}_{j_s} \pm 1, \mathbf{core}_{(\nu - s)}(\tau))$, where the sign equals $\text{sign}(r)$.

Then there are only two distinct Verma objects which map non-trivially into $M(\tau)$: $M(\mu)$ and $M(\tau)$ itself (see Subsection 10.3). Both map uniquely (up to scalar multiple) into $M(\tau)$.

Lemma 11.4.1. *The image of $M(\mu)$ in $M(\tau)$ is a simple $O_{c, \nu}$ -object.*

Proof. Assume X_{λ} is a simple singular $\text{Rep}(S_{\nu})$ -subobject in $\text{Im}(M(\mu))$, and $X_{\lambda} \subset M(\tau)$ lies in degree m' . Then the action of \mathbf{h} on $M(\tau)$ and $J(\tau)/\text{Im}(M(\mu))$ gives (see Proposition 7.1.1):

$$\begin{aligned} c' &= \frac{\nu - s}{r} \\ \frac{|\tau|^2 - |\lambda|^2 - (|\tau| - |\lambda|)}{2} + ct(\tau) - ct(\lambda) &= c'm' + (|\tau| - |\lambda|)\nu \end{aligned}$$

By assumption, we get: $\mu = \lambda$, which implies $m = m'$, and since X_{λ} is a simple singular $\text{Rep}(S_{\nu})$ -subobject in $\text{Im}(M(\mu))$, this means that X_{λ} coincides with the lowest weight in $\text{Im}(M(\mu))$.

Thus $\text{Im}(M(\mu))$ is simple. \square

We now want to show that there is a short exact sequence:

$$0 \longrightarrow L(\mu) \longrightarrow M(\tau) \longrightarrow L(\tau) \longrightarrow 0$$

This is equivalent to saying that $\text{Im}(M(\mu)) = J(\tau)$.

Lemma 11.4.2. $M(\tau)/\text{Im}(M(\mu))$ is a simple $O_{c,\nu}$ -object.

Proof. Assume X_λ is a simple singular $\text{Rep}(S_\nu)$ -subobject in $M(\tau)/\text{Im}(M(\mu))$, and its preimage $X_\lambda \subset M(\tau)$ lies in degree m' .

Then the action of \mathbf{h} on $M(\tau)$ and $J(\tau)/\text{Im}(M(\mu))$ gives (see Proposition 7.1.1):

$$c' = \frac{\nu - s}{r}$$

$$\frac{|\tau|^2 - |\lambda|^2 - (|\tau| - |\lambda|)}{2} + ct(\tau) - ct(\lambda) = c'm' + (|\tau| - |\lambda|)\nu$$

We now need to consider separately the case when $\lambda = \mu$. Indeed, in that case the above equations mean that $m' = m$, and thus we have $X := X_\mu \subset M(\tau)$ (not a singular subobject) which lies in degree m' and whose image in $M(\tau)/\text{Im}(M(\mu))$ is not 0 and singular.

The former implies that $y_{M(\tau)}(\mathbf{h}_0 \otimes X) \neq 0$ is a direct sum of simple $\text{Rep}(S_\nu)$ -objects lying in degree $m - 1$ of $M(\tau)$.

The fact that image of X in $M(\tau)/\text{Im}(M(\mu))$ is singular would mean that

$$y_{M(\tau)}(\mathbf{h}_0 \otimes X) \subset \text{Im}(M(\mu))$$

But this leads to a contradiction, since $\text{Im}(M(\mu))$ lies in degrees strictly higher than $m - 1$ of $M(\tau)$.

So $\lambda \neq \mu$, and this contradicts our assumption. \square

Thus for generic (c', ν) such that $c' = \frac{\nu-s}{r}$, we have a short exact sequence:

$$0 \longrightarrow L(\mu) \longrightarrow M(\tau) \longrightarrow L(\tau) \longrightarrow 0$$

and a long exact sequence

$$\begin{aligned} \dots &\longrightarrow M(\mathbf{rec}(\tilde{\tau}_{j_s} \pm l, \mathbf{core}_{(\nu-s)}(\tau))) \longrightarrow M(\mathbf{rec}(\tilde{\tau}_{j_s} \pm (l-1), \mathbf{core}_{(\nu-s)}(\tau))) \longrightarrow \dots \\ \dots &\longrightarrow M(\mathbf{rec}(\tilde{\tau}_{j_s} \pm 1, \mathbf{core}_{(\nu-s)}(\tau))) \longrightarrow M(\tau) \longrightarrow L(\tau) \longrightarrow 0 \end{aligned}$$

As before, the sign corresponds to the sign of r , and this sequence ends (on the left) with $M(\mathbf{rec}(0, \mathbf{core}_{(\nu-s)}(\tau)))$ if $r < 0$.

This allows us to compute the character of $L(\tau)$ by Euler formula:

$$chL(\tau) = \sum_{l \in \mathbb{Z}_+, \text{ and } l \leq \tilde{\tau}_{j_s} \text{ if } r < 0} (-1)^l chM(\mathbf{rec}(\tilde{\tau}_{j_s} \pm l, \mathbf{core}_{(\nu-s)}(\tau)))$$

Using Subsection 11.2 and the fact that

$$h_{c,\nu}(\mathbf{rec}(\tilde{\tau}_{j_s} \pm l, \mathbf{core}_{(\nu-s)}(\tau))) = h_{c,\nu}(\tau) + c(\nu - s)(j_s - k_{s,l} \pm l)$$

get:

$$(2) \quad chL(\tau) = \frac{t^{h_{c,\nu}(\tau)} \left(\sum_{l \in \mathbb{Z}_+, \text{ and } l \leq \tilde{\tau}_{j_s} \text{ if } r < 0} (-1)^l X_{\mathbf{rec}(\tilde{\tau}_{j_s} \pm l, \mathbf{core}_{(\nu-s)}(\tau))} t^{c(\nu-s)(j_s - k_{s,l} \pm l)} \right)}{\sum_{n \geq 0} (-1)^n \Lambda^n \mathbf{h}_0 t^n}$$

As before, the sign corresponds to the sign of r .

Example 11.4.3. Let $c\nu = 1, \text{Re}(c) > 0$.

If X_λ lies in degree m' of $M(\pi^n)$ is a singular $\text{Rep}(S_\nu)$ -subobject (π^n is a column diagram with n cells), then Equation (1) gives us:

$$m = (|\lambda| - n)c\nu - f(\lambda)c = (|\lambda| - n) - f(\lambda)c$$

and Pieri's rule (Proposition 4.2.1) implies that $m \geq ||\lambda| - n|$. Since $m, |\lambda|, f(\lambda) \in \mathbb{Z}$, we get that $c \in \mathbb{Q}_{>0}$.

We also have: $f(\lambda) \geq 0$, with equality iff λ is a column diagram (see Lemma 7.2.1). So

$$||\lambda| - n| \leq m = (|\lambda| - n) - f(\lambda)c \leq (|\lambda| - n)$$

which means that $f(\lambda) = 0, |\lambda| \geq n$, i.e. $\lambda = \pi^k$ for some $k \geq n$ (in fact, $k > n$), and X_{π^k} lies in degree $m = k - n$ of $M(\pi^n)$.

But by Pieri's rule (Proposition 4.2.1), for $k > 1$, $T^k \mathfrak{h}_0$ (k -th tensor power of \mathfrak{h}_0) contains only one copy of $X_{\pi^k} \cong \Lambda^k(\mathfrak{h}_0)$, and it does not lie inside $\text{Sym}^k(\mathfrak{h}_0)$. So $\text{Sym}^k(\mathfrak{h}_0)$ doesn't contain subobjects isomorphic to X_{π^k} .

So $M(\pi^n)$ has only one non-trivial singular $\text{Rep}(S_\nu)$ -subobject: $X_{\pi^{n+1}}$, which lies in degree 1 of $M(\pi^n)$.

In this case, we can compute the character of $L(\pi^n)$ as in Equation (2). We assumed that $c\nu = 1$, so $h_{c,\nu}(\pi^n) = \frac{(\nu-1)(1-c\nu)}{2} + n = n$, hence

$$chL(\pi^n) = \frac{\left(\sum_{l \in \mathbb{Z}_+} (-1)^l X_{\pi^{n+l}} t^{n+l}\right)}{\sum_{l \in \mathbb{Z}_+} (-1)^l \Lambda^l \mathfrak{h}_0 t^l} = \frac{\left(\sum_{l \in \mathbb{Z}_+} (-1)^l \Lambda^{n+l} \mathfrak{h}_0 t^{n+l}\right)}{\sum_{l \in \mathbb{Z}_+} (-1)^l \Lambda^l \mathfrak{h}_0 t^l}$$

But $\Lambda^{n+l} \mathfrak{h}_0 \cong X_{\pi^{n+l}}$ in $\text{Rep}(S_\nu)$, so

$$chL(\pi^n) = \frac{\sum_{l \in \mathbb{Z}_+} (-1)^l \Lambda^{n+l} \mathfrak{h}_0 t^{n+l}}{\sum_{l \in \mathbb{Z}_+} (-1)^l \Lambda^l \mathfrak{h}_0 t^l}$$

Note that $L(\emptyset) \cong X_\emptyset = 1$ as $\text{Rep}(S_\nu)$ (ind)-objects, with maps $x_{L(\emptyset)}, y_{L(\emptyset)} = 0$.

Example 11.4.4. Similarly, for a generic point (c, ν) on the line $c\nu = k$, $k \in \mathbb{Z}_{>0}$, we have: $s = 0, r = k$ and so

$$\text{rec}(\tilde{\tau}_{j_s} \pm l, \text{core}_{(\nu-s)}(\pi^n)) = X_{\pi^{n+l}}$$

$$chL(\pi^n) = t^{kn - \frac{(\nu-1)(k-1)}{2}} \cdot \frac{\sum_{l \in \mathbb{Z}_+} (-1)^l \Lambda^{n+l} \mathfrak{h}_0 t^{kl}}{\sum_{l \geq 0} (-1)^l \Lambda^l \mathfrak{h}_0 t^l}$$

and in particular

$$chL(\emptyset) = t^{-\frac{(\nu-1)(k-1)}{2}} \cdot \frac{\sum_{l \in \mathbb{Z}_+} (-1)^l \Lambda^l \mathfrak{h}_0 t^{kl}}{\sum_{l \geq 0} (-1)^l \Lambda^l \mathfrak{h}_0 t^l} = t^{-\frac{(\nu-1)(k-1)}{2}} \cdot \frac{ch_t S \mathfrak{h}_0}{ch_{t^k} S \mathfrak{h}_0}$$

(see [2] or [9, Corollary 3.50] for the corresponding result for $O(H_c(n))$).

As in the case when $k = 1$, we can also compute the character of $L_{c,\nu}(\emptyset)$ explicitly:

Proposition 11.4.5. Let (c, ν) be a generic point on the line $c\nu = k$, where $k \in \mathbb{Z}_{>0}$ is fixed. Then

$$ch_q \text{Hom}_{S_\nu}(X_\mu, L(\emptyset)) = \frac{q^{|\mu|} s_\mu(1, q, q^2, \dots, q^{k-2})}{\prod_{2 \leq j \leq k} (1 - q^j)}$$

Proof. From the exact sequence

$$\begin{aligned} \dots &\longrightarrow M(\mathbf{rec}(\pi^l)) \longrightarrow M(\pi^{l-1}) \longrightarrow \dots \\ \dots &\longrightarrow M(\pi^1) \longrightarrow M(\emptyset) \longrightarrow L(\emptyset) \longrightarrow 0 \end{aligned}$$

and the character formula Corollary 11.3.4 for $Hom(X_\mu, S\mathfrak{h}_0 \otimes X_{\pi^l})$, we have:

$$ch_q Hom_{S_\nu}(X_\mu, L(\emptyset)) = (1-q) \sum_{l \geq 0} \sum_{\lambda \text{ is a partition}} (-1)^l q^{kl} \bar{\gamma}_{\Lambda^l \mathfrak{h}_0, \mu}^\lambda \frac{q^{|\lambda|} s_\lambda(1, q, q^2, \dots)}{\prod_{j \geq 1} (1 - q^j)}$$

One immediately sees that this is the q -adic limit, as n tends to infinity, of the sequence of formal power series

$$P_n(q) := (1-q) \sum_{l \geq 0} \sum_{\lambda \text{ is a partition of } n} (-1)^l q^{kl} \gamma_{\Lambda^l \mathfrak{h}_0, \tilde{\mu}(n)}^\lambda s_\lambda(1, q, q^2, \dots)$$

where \mathfrak{h}_0 is the reflection representation of S_n .

We now give a positive formula for $P_n(q)$. First, recall that (see [17, Chapter I, Par. 7])

$$\gamma_{\Lambda^l \mathfrak{h}_0, \tilde{\mu}(n)}^\lambda = \frac{1}{n!} \sum_{w \in S_n} \chi^{\Lambda^l \mathfrak{h}_0}(w) \chi^{\tilde{\mu}(n)}(w) \chi^\lambda(w)$$

(here $\chi^\beta(w) = tr|_\beta(w)$ is the value at w of the character of S_n corresponding to the irreducible representation β).

Now, from the exact sequence

$$0 \rightarrow \Lambda^{n-1} \mathfrak{h}_0 \otimes S\mathfrak{h}_0 \dots \rightarrow \Lambda^m \mathfrak{h}_0 \otimes S\mathfrak{h}_0 \rightarrow \Lambda^{m-1} \mathfrak{h}_0 \otimes S\mathfrak{h}_0 \rightarrow \dots \rightarrow \mathfrak{h}_0 \otimes S\mathfrak{h}_0 \rightarrow S\mathfrak{h}_0 \rightarrow \mathbb{C} \rightarrow 0$$

we have:

$$\begin{aligned} \sum_{l \geq 0} (-1)^l q^{kl} \chi^{\Lambda^l \mathfrak{h}_0}(w) &= \frac{1}{\sum_{l \geq 0} q^{kl} tr|_{S^l \mathfrak{h}_0}(w)} = \dots \\ \dots &= \frac{1}{\sum_{l \geq 0} q^{kl} (1 - q^k) tr|_{S^l \mathfrak{h}}(w)} = \frac{1}{1 - q^k} \prod_{1 \leq j \leq l(\rho(w))} (1 - q^{k\rho(w)_j}) \end{aligned}$$

(here $\rho(w)$ is the cycle type of w).

On the other hand,

$$\sum_{\lambda \text{ is a partition of } n} \chi^\lambda(w) s_\lambda(1, q, q^2, \dots) = \prod_{1 \leq j \leq l(\rho(w))} \frac{1}{1 - q^{\rho(w)_j}}$$

Thus we obtain:

$$\begin{aligned} P_n(q) &= (1-q) \sum_{l \geq 0} \sum_{\lambda \text{ is a partition of } n} (-1)^l q^{kl} \gamma_{\Lambda^l \mathfrak{h}_0, \tilde{\mu}(n)}^\lambda s_\lambda(1, q, q^2, \dots) = \dots \\ \dots &= \frac{1-q}{1-q^k} \frac{1}{n!} \sum_{w \in S_n} \chi^{\tilde{\mu}(n)}(w) \prod_{1 \leq j \leq l(\rho(w))} \frac{1 - q^{k\rho(w)_j}}{1 - q^{\rho(w)_j}} = \dots \\ \dots &= \frac{1-q}{1-q^k} \frac{1}{n!} \sum_{w \in S_n} \chi^{\tilde{\mu}(n)}(w) \prod_{1 \leq j \leq l(\rho(w))} (1 + q + q^2 + \dots + q^{k\rho(w)_j}) = \frac{1-q}{1-q^k} s_{\tilde{\mu}(n)}(1, q, \dots, q^{k-1}) \end{aligned}$$

Taking the q -adic limit when $n \rightarrow \infty$, we obtain: the q -adic limit of $s_{\tilde{\mu}(n)}(1, q, \dots, q^{k-1})$ is $\frac{q^{|\mu|} s_{\mu}(1, q, q^2, \dots, q^{k-2})}{\prod_{1 \leq j \leq k-1} (1 - q^j)}$, and thus

$$ch_q \operatorname{Hom}_{S_{\nu}}(X_{\mu}, L(\emptyset)) = \frac{q^{|\mu|} s_{\mu}(1, q, q^2, \dots, q^{k-2})}{\prod_{2 \leq j \leq k} (1 - q^j)}$$

□

Remark 11.4.6. Note that $s_{\mu}(1, q, q^2, \dots, q^{k-2}) = 0$ if $l(\mu) \geq k$, so $\operatorname{Hom}_{S_{\nu}}(X_{\mu}, L(\emptyset)) = 0$ whenever $l(\mu) \geq k$.

12. LENGTH OF VERMA OBJECTS

In this section, we will discuss the set of points (c, ν) such that the Verma object $M_{c,\nu}(\tau)$ is of finite length. We will prove the following theorem:

Theorem 12.0.7. *For any $c \notin \mathbb{Q}_{<0}$, the Verma object $M_{c,\nu}(\tau)$ is of finite length.*

By Subsection 10.2, $M_{c,\nu}(\tau)$ has infinite length whenever (c, ν) is the intersection point of infinitely many curves of form $\frac{1}{c} = \frac{\nu-s}{r}$ where $s \in C_{\tau}, r \in \mathbb{Z} \setminus 0$.

For instance, when $\tau = \emptyset$, we have:

Lemma 12.0.8. *For any $c \in \mathbb{Q}_{<0}$ there exists $\nu \in \mathbb{Q}$ such that the Verma object $M_{c,\nu}(\emptyset)$ has infinite length.*

Proof. Indeed, for any $c \in \mathbb{Q}_{<0}$ and any $\nu \in \mathbb{Q}$ such that $\operatorname{den}(\nu) \mid \operatorname{num}(c), \operatorname{den}(c) \mid (\operatorname{num}(\nu) + \operatorname{den}(\nu))$ (i.e. $\operatorname{num}(\nu)\operatorname{den}(\nu)^{-1} \equiv -1 \pmod{\operatorname{den}(c)}$), we have:

$$\{s \in \mathbb{Z}_+ \mid c(\nu - s) \in \mathbb{Z}_{>0}\} = \{s \in \mathbb{Z}_+ \mid s \equiv -1 \pmod{\operatorname{den}(c)}, s > \nu\}$$

Now, recall that $C_{\emptyset} = \mathbb{Z}_+$. So any s such that $s \equiv -1 \pmod{\operatorname{den}(c)}, s > \nu$ gives us a non-trivial morphism $M_{c,\nu}(\tau^s) \rightarrow M_{c,\nu}(\emptyset)$. The image of each of these morphisms has a simple object $L_{c,\nu}(\tau^s)$ as a quotient, and since all the weights τ^s are different, each of these simple objects $L_{c,\nu}(\tau^s)$ contributes to the length of $M_{c,\nu}(\emptyset)$. This proves that $M_{c,\nu}(\emptyset)$ has infinite length. □

The proof consists of two parts: proving that whenever $c > 0$, the Verma object $M_{c,\nu}(\tau)$ is of finite length (we will prove this statement in Subsection 12.2), and proving that whenever $c \notin \mathbb{Q}$, the Verma object $M_{c,\nu}(\tau)$ is of finite length, too (we will prove that in Subsection 12.3).

Remark 12.0.9. In the classical case, all modules in $O(H_c(n))$ have finite length. See [9, Corollary 3.26].

12.1. Bounds on the graded space $\operatorname{Hom}_{\operatorname{Rep}(S_{\nu})}(X_{\mu}, S\mathfrak{h}_0 \otimes X_{\tau})$. In this subsection, we give a bound on the least degree in which a simple $\operatorname{Rep}(S_n u)$ object X_{μ} can lie in the $\operatorname{Rep}(S_n u)$ ind-object $S\mathfrak{h}_0 \otimes X_{\tau}$.

Recall that we have the following corollary of Lemma ??.

Corollary 12.1.1. *In particular, for $\tau = \emptyset$, we have:*

$$ch_q(\operatorname{Hom}_{S_n}(\tilde{\mu}(n), \mathbb{C}[x_1, \dots, x_n])) = s_{\tilde{\mu}(n)}(1, q, q^2, \dots)$$

By the definition of a Schur symmetric function, $s_{\tilde{\mu}(n)}(1, q, q^2, \dots)$ is divisible by $q^{\sum_k \mu_k k}$. So for any $n \gg 0$, the minimal degree of $S\mathfrak{h} = \mathbb{C}[x_1, \dots, x_n]$ in which $\tilde{\mu}(n)$ appears is greater than or equal $\sum_k \mu_k k$. By the description of Deligne's category, this implies:

Corollary 12.1.2. *For any ν , we have, in $\text{Rep}(S_\nu)$: the minimal degree of $S\mathfrak{h}$ (and thus of $S\mathfrak{h}_0$ as well) in which X_μ appears is greater than or equal $\sum_k \mu_k k$.*

We now generalize this result to the following:

Proposition 12.1.3. *Consider the \mathbb{Z}_+ -graded complex vector space $\text{Hom}_{\text{Rep}(S_\nu)}(X_\mu, S\mathfrak{h}_0 \otimes X_\tau)$ (the grading inherited from $S\mathfrak{h}_0$). Let m be a grade of this space containing a non-zero morphism. Then*

$$m \geq \sum_{k \geq 1} \mu_k k - l(\mu) |\tau| - \frac{3}{2} |\tau|^2 - \frac{|\tau|}{2}$$

Proof. Since any object of $\text{Rep}(S_\nu)$ is isomorphic to its dual, $\text{Hom}_{\text{Rep}(S_\nu)}(X_\mu, S\mathfrak{h}_0 \otimes X_\tau) = \text{Hom}_{\text{Rep}(S_\nu)}(X_\mu \otimes X_\tau, S\mathfrak{h}_0)$ as \mathbb{Z}_+ -graded vector spaces.

By the definition of m , this means that $X_\mu \otimes X_\tau$ and $S^m \mathfrak{h}_0$ have a common composition factor. Now, $X_\tau \subset \mathfrak{h}_0^{\otimes |\tau|}$, so $X_\mu \otimes \mathfrak{h}_0^{\otimes |\tau|}$ and $S^m \mathfrak{h}_0$ have a common composition factor.

A simple subobject X_λ of $X_\mu \otimes \mathfrak{h}_0^{\otimes |\tau|}$ satisfies the condition arising from Pieri's rule (Proposition 4.2.1):

Condition 12.1.4. λ can be obtained from μ by performing at most $|\tau|$ steps, each consisting of either adding a cell, deleting a cell or moving a cell (i.e. deleting a cell and then adding a cell).

We know that for some λ as above, $X_\lambda \subset S^m \mathfrak{h}_0^*$, so by Corollary 12.1.2,

$$m \geq \sum_k \lambda_k k$$

We are left with the problem of giving a bound for $|\sum_{k \geq 1} \lambda_k k - \sum_{k \geq 1} \mu_k k|$ for two Young diagrams satisfying Condition 12.1.4. Note that this problem is symmetric in μ, λ .

By the condition Condition 12.1.4, $\sum_{k \geq 1} \lambda_k k$ is maximal when λ is obtained from μ by adding a column of $|\tau|$ cells to μ , i.e.

$$\lambda_i := \begin{cases} \mu_i & \text{if } 1 \leq i \leq l(\mu) \\ 1 & \text{if } l(\mu) + 1 \leq i \leq l(\mu) + |\tau| \end{cases}$$

So

$$\begin{aligned} \sum_{k \geq 1} \lambda_k k &\leq \sum_{k \geq 1} \mu_k k + (l(\mu) + 1) + (l(\mu) + 2) + \dots + (l(\mu) + |\tau|) = \dots \\ \dots &= \sum_{k \geq 1} \mu_k k + l(\mu) |\tau| + \frac{|\tau| (|\tau| + 1)}{2} \end{aligned}$$

and so

$$\begin{aligned} \sum_{k \geq 1} \mu_k k &\leq \sum_{k \geq 1} \lambda_k k + l(\lambda) |\tau| + \frac{|\tau| (|\tau| + 1)}{2} \leq \dots \\ \dots &\leq \sum_{k \geq 1} \lambda_k k + l(\mu) |\tau| + |\tau|^2 + \frac{|\tau| (|\tau| + 1)}{2} \end{aligned}$$

Thus

$$\sum_{k \geq 1} \lambda_k k \geq \sum_{k \geq 1} \mu_k k - l(\mu) |\tau| - |\tau|^2 - \frac{|\tau| (|\tau| + 1)}{2}$$

and, finally, we get:

$$\begin{aligned} m &\geq \sum_k \lambda_k k \geq \sum_{k \geq 1} \mu_k k - l(\mu) |\tau| - |\tau|^2 - \frac{|\tau|(|\tau| + 1)}{2} = \dots \\ \dots &= \sum_{k \geq 1} \mu_k k - l(\mu) |\tau| - \frac{3}{2} |\tau|^2 - \frac{|\tau|}{2} \end{aligned}$$

□

We will use Proposition 12.1.3 in several instances, but here is a simple application of its special case Corollary 12.1.2:

Corollary 12.1.5. *Let $c, \nu \in \mathbb{R}, c > 0$. Let X_μ be a simple $\text{Rep}(S_\nu)$ -object lying in degree $m > 0$ of $M_{c,\nu}(\emptyset)$ and assume Equation (7.1.1) holds for c, ν (this happens, for instance, if X_μ is a simple singular $\text{Rep}(S_\nu)$ -subobject of a composition factor of $M_{c,\nu}(\emptyset)$). Then $2 \max\{1, c\} \nu - 1 > |\mu|$.*

Proof. Equation (7.1.1) for $\tau = \emptyset$, together with Proposition 12.1.2, give us:

$$m = c\nu |\mu| - \frac{c}{2} |\mu|^2 + \frac{c}{2} |\mu| - c \cdot ct(\mu) \geq \sum_k \mu_k k$$

i.e.

$$-\frac{c}{2} |\mu|^2 + \left(\frac{c}{2} + c\nu\right) |\mu| + (c - 1) \sum_k \mu_k k - c \sum_k \check{\mu}_k k \geq 0$$

But $0 < \sum_k \mu_k k \leq \frac{1}{2} |\mu|^2 + \frac{1}{2} |\mu|$ and $\sum_k \check{\mu}_k k \geq |\mu| > 0$ (in both cases equalities occur when μ is a column Young diagram).

If $c \geq 1$, we get:

$$-\frac{1}{2} |\mu|^2 + \left(-\frac{1}{2} + c\nu\right) |\mu| \geq 0$$

and hence

$$2c\nu - 1 \geq |\mu|.$$

If $0 \leq c \leq 1$, then $(c - 1) \sum_k \mu_k k - c \sum_k \check{\mu}_k k < 0$, so $-\frac{c}{2} |\mu|^2 + \left(\frac{c}{2} + c\nu\right) |\mu| > 0$ and thus

$$|\mu| < 1 + 2\nu$$

□

12.2. Length of Verma objects for $c \in \mathbb{R}_{>0}$.

Proposition 12.2.1. *For any $c \in \mathbb{R}_{>0}$, the Verma object $M_{c,\nu}(\tau)$ is of finite length.*

Proof. Fix c, ν such that $c \in \mathbb{R}_{>0}$. We need to prove that $M(\tau)$ has only finitely many composition factors. By Proposition 7.1.1, it is enough to prove that there exist only finitely a Young diagrams μ such that X_μ lies in degree m of $M(\tau)$ and

$$m = c\nu(|\mu| - |\tau|) + c \left(\frac{|\tau|^2 - |\mu|^2 - (|\tau| - |\mu|)}{2} + ct(\tau) - ct(\mu) \right)$$

By Corollary 12.1.3, it is, in fact, enough to prove that

$$c\nu(|\mu| - |\tau|) + c \left(\frac{|\tau|^2 - |\mu|^2 - (|\tau| - |\mu|)}{2} + ct(\tau) - ct(\mu) \right) < \sum_{k \geq 1} \mu_k k - l(\mu) |\tau| - \frac{3}{2} |\tau|^2 - \frac{|\tau|}{2}$$

for all but finitely many Young diagrams μ .

Recall that $ct(\mu) = \sum_{k \geq 1} \check{\mu}_k k - \sum_{k \geq 1} \mu_k k$, so subtracting LHS from RHS in the above expression, we get

$$\begin{aligned}
& -c\nu |\mu| + c\nu |\tau| - cf(\tau) + c \frac{|\mu|^2 - |\mu|}{2} + c \sum_{k \geq 1} \check{\mu}_k k - c \sum_{k \geq 1} \mu_k k + \dots \\
& \dots + \sum_{k \geq 1} \mu_k k - l(\mu) |\tau| - \frac{3}{2} |\tau|^2 - \frac{|\tau|}{2} = c \frac{|\mu|^2 - |\mu|}{2} - \dots \\
& \dots - c\nu |\mu| + c \sum_{k \geq 1} \check{\mu}_k k - (c-1) \sum_{k \geq 1} \mu_k k - l(\mu) |\tau| - \frac{3}{2} |\tau|^2 - \frac{|\tau|}{2} + c\nu |\tau| - cf(\tau)
\end{aligned}$$

where $f(\tau) := \frac{|\tau|^2 - |\tau|}{2} + ct(\tau)$.

So we need to show that for all but finitely many Young diagrams μ ,

$$c \frac{|\mu|^2 - |\mu|}{2} - c\nu |\mu| + c \sum_{k \geq 1} \check{\mu}_k k - (c-1) \sum_{k \geq 1} \mu_k k - l(\mu) |\tau| - \frac{3}{2} |\tau|^2 - \frac{|\tau|}{2} + c\nu |\tau| - cf(\tau) > 0$$

Since $l(\mu) \leq |\mu|$, it is in fact enough to check that all but finitely many Young diagrams μ satisfy the following condition:

Condition 12.2.2.

$$\frac{c}{2} |\mu|^2 - \frac{c}{2} |\mu| - c\nu |\mu| - |\mu| |\tau| + c \sum_{k \geq 1} \check{\mu}_k k - (c-1) \sum_{k \geq 1} \mu_k k - \frac{3}{2} |\tau|^2 - \frac{|\tau|}{2} + c\nu |\tau| - cf(\tau) > 0$$

We now have to consider two cases separately:

- If $0 < c < 1$, then $c \sum_{k \geq 1} \check{\mu}_k k - (c-1) \sum_{k \geq 1} \mu_k k > 0$, and

$$\frac{c}{2} |\mu|^2 - \frac{c}{2} |\mu| - c\nu |\mu| - |\mu| |\tau| - \frac{3}{2} |\tau|^2 - \frac{|\tau|}{2} + c\nu |\tau| - cf(\tau) > 0$$

for μ such that $|\mu|$ is large enough.

- If $c \geq 1$, then using the facts that $\sum_{k \geq 1} \check{\mu}_k k \geq |\mu|$ and $\sum_{k \geq 1} \mu_k k \leq \frac{|\mu|(|\mu|+1)}{2}$, we get:

$$\begin{aligned}
& \frac{c}{2} |\mu|^2 - \frac{c}{2} |\mu| - c\nu |\mu| - |\mu| |\tau| + c \sum_{k \geq 1} \check{\mu}_k k - (c-1) \sum_{k \geq 1} \mu_k k \geq \dots \\
& \dots \geq \frac{c}{2} |\mu|^2 - \frac{c}{2} |\mu| - c\nu |\mu| - |\mu| |\tau| + c |\mu| - (c-1) \frac{|\mu|(|\mu|+1)}{2} = \dots \\
& \dots = \frac{1}{2} |\mu|^2 + \left(\frac{1}{2} - c\nu - |\tau| \right) |\mu|
\end{aligned}$$

So the RHS of Equation (12.2.2) is at least

$$\frac{1}{2} |\mu|^2 + \left(\frac{1}{2} - c\nu - |\tau| \right) |\mu| - \frac{3}{2} |\tau|^2 - \frac{|\tau|}{2} + c\nu |\tau| - cf(\tau)$$

which is obviously positive whenever $|\mu|$ is large enough, i.e. for all but finitely many Young diagrams μ . □

12.3. Length of Verma objects for $c \notin \mathbb{Q}$.

Proposition 12.3.1. *For any $c \notin \mathbb{Q}$, the Verma object $M_{c,\nu}(\tau)$ is of finite length.*

Proof. Assume that for some $c \notin \mathbb{Q}, \nu$, a Verma object $M_{c,\nu}(\tau)$ has infinite length.

$M(\tau)$ has infinitely many composition factors; in each composition factor M_j , one can fix a simple singular $\text{Rep}(S_\nu)$ -subobject X_{μ^j} . It would exist because the set of eigenvalues of \mathbf{h} on M_j lies inside $\mathbb{Z}_+ + h_{c,\nu}(\tau)$ and thus has a minimal element, and a simple $\text{Rep}(S_\nu)$ -subobject corresponding to the minimal eigenvalue would be singular.

Since M_j is a composition factor of $M(\tau)$, there exists a positive integer m^j such that $h_{c,\nu}(\mu^j) + m^j = h_{c,\nu}(\tau)$, and $X_{\mu^j} \subset S^{m^j} \mathfrak{h}_0 \otimes \tau$. Similarly to the proof of Proposition 7.1.1, for any j we have:

$$(1/c, \nu) \in \mathcal{L}_{\tau, \mu^j, m^j}$$

Since $(1/c, \nu) \in \mathcal{L}_{\tau, \mu^j, m^j}$ for any j , and $c \notin \mathbb{Q}$, Subsection 10.3 implies that for all j the lines $\mathcal{L}_{\tau, \mu^j, m^j}$ coincide; that is, the equations

$$\frac{1}{c} m^j + \nu(|\tau| - |\mu^j|) = \frac{|\tau|^2 - |\mu^j|^2 - (|\tau| - |\mu^j|)}{2} + ct(\tau) - ct(\mu^j)$$

define the same line for all j .

This means that there exist constants $C, C' \in \mathbb{Q}$ such that for any j ,

$$\begin{aligned} \frac{1}{|\mu^j| - |\tau|} \left(\frac{|\tau|^2 - |\mu^j|^2 - (|\tau| - |\mu^j|)}{2} + ct(\tau) - ct(\mu^j) \right) &= C' \\ \text{and } \frac{|\mu^j| - |\tau|}{m^j} &= C \end{aligned}$$

Since $X_{\mu^j} \subset S^{m^j} \mathfrak{h}_0 \otimes \tau$, Pieri's rule (Proposition 4.2.1) implies that $|C| \leq 1$, and Corollary 12.1.3 implies that for any j ,

$$m^j \geq \sum_{k \geq 1} (\mu^j)_k k - l(\mu^j) |\tau| - \frac{3}{2} |\tau|^2 - \frac{|\tau|}{2}$$

So for any j , the following condition hold for $\mu := \mu^j$:

Condition 12.3.2.

$$\begin{aligned} \frac{1}{|\mu| - |\tau|} \left(\frac{|\mu|^2 - |\mu|}{2} + ct(\mu) - f(\tau) \right) &= C' \\ \frac{|\mu| - |\tau|}{m^j} &= C \\ m^j &\geq \sum_{k \geq 1} (\mu)_k k - l(\mu) |\tau| - \frac{3}{2} |\tau|^2 - \frac{|\tau|}{2} \end{aligned}$$

with C, C' - constants, $C \leq 1$, and $f(\tau) = \frac{|\tau|^2 - |\tau|}{2} + ct(\tau)$ as in Subsection 7.2.

We will show that for given C, C' as above, Condition 12.3.2 can only hold for a finite number of Young diagrams μ .

The second and third parts of Condition 12.3.2 give:

$$C(|\mu| - |\tau|) \geq \sum_{k \geq 1} \mu_k k - l(\mu) |\tau| - \frac{3}{2} |\tau|^2 - \frac{|\tau|}{2}$$

i.e.

$$\frac{3}{2} |\tau|^2 + \frac{|\tau|}{2} - C |\tau| \geq \sum_{k \geq 1} \mu_k k - C |\mu| - l(\mu) |\tau|$$

But

$$\begin{aligned} \sum_{k \geq 1} \mu_k k - C |\mu| &= \sum_{k \geq 1} (\mu_k - 1)k + \frac{l(\mu)^2 - l(\mu)}{2} - Cl(\mu) - C(|\mu| - l(\mu)) = \dots \\ \dots &= \sum_{k \geq 1} (\mu_k - 1)(k - C) + \frac{l(\mu)^2 - l(\mu)}{2} - Cl(\mu) \end{aligned}$$

So

$$\frac{3}{2} |\tau|^2 + \frac{|\tau|}{2} - C |\tau| \geq \sum_{k \geq 1} (\mu_k - 1)(k - C) + \frac{l(\mu)^2}{2} - l(\mu) \left(\frac{1}{2} + C + |\tau| \right)$$

The summands in the expression $\sum_{k \geq 1} (\mu_k - 1)(k - C)$ are non-negative. If $C < 1$, then $k - C > 0$ for any $k \in \{1, \dots, l(\mu)\}$, and this means that $\sum_{k \geq 1} (\mu_k - 1)$ and $l(\mu)$ are bounded (in this proof, “bounded” means bounded by some functions of $|\tau|, C, C'$). So their sum, $|\mu|$, is bounded as well, and so Condition 12.3.2 is only satisfied by a finite number of Young diagrams μ .

If $C = 1$, then $k - C > 0$ for any $k \in \{2, \dots, l(\mu)\}$, and this means that $\sum_{k \geq 2} (\mu_k - 1)$ and $l(\mu)$ are bounded, which implies that $|\mu| - \mu_1$ is bounded.

We would like to prove that $|\mu|$ is bounded, which would imply that Condition 12.3.2 is only satisfied by a finite number of Young diagrams μ . Since $|\mu| - \mu_1$ is bounded, it only remains to show that μ_1 is bounded.

For this, we use the first part of Condition 12.3.2:

$$\frac{|\mu|^2 - |\mu|}{2} + ct(\mu) - f(\tau) = C'(|\mu| - |\tau|)$$

Denote by $\bar{\mu}$ the Young diagram obtained by removing the top row of μ (so $|\bar{\mu}| = |\mu| - \mu_1$). We know that the size of $\bar{\mu}$ is bounded, i.e. there are only a finite number of possibilities for $\bar{\mu}$, so $f(\bar{\mu})$ is bounded as well.

We write $f(\mu) = \frac{|\mu|^2 - |\mu|}{2} + ct(\mu)$ in terms of $\mu_1, \bar{\mu}$:

$$\begin{aligned} ct(\mu) &= \frac{\mu_1^2}{2} - \frac{\mu_1}{2} + ct(\bar{\mu}) \\ \frac{|\mu|^2 - |\mu|}{2} &= \frac{\mu_1^2}{2} - \frac{\mu_1}{2} + \mu_1 |\bar{\mu}| + \frac{|\bar{\mu}|^2 - |\bar{\mu}|}{2} \\ \text{so } f(\mu) &= \frac{|\mu|^2 - |\mu|}{2} + ct(\mu) = \mu_1^2 - \mu_1 + \mu_1 |\bar{\mu}| + f(\bar{\mu}) \end{aligned}$$

Thus

$$\mu_1^2 - \mu_1 + \mu_1 |\bar{\mu}| + f(\bar{\mu}) - f(\tau) = C'(|\mu| - |\tau|)$$

i.e.

$$\mu_1^2 + (|\bar{\mu}| - 1 - C')\mu_1 + f(\bar{\mu}) - |\bar{\mu}| - f(\tau) + C' |\tau| = 0$$

We know that $|\bar{\mu}|, f(\bar{\mu})$ are bounded, so for μ_1 large enough, the LHS of the last equality will be positive. So μ_1 is bounded, and this completes the proof. \square

REFERENCES

- [1] R. Bezrukavnikov, P. Etingof, *Parabolic induction and restriction functors for rational Cherednik algebras*, Selecta Math. (N.S.) 14 (2009), 397-425, arXiv:0803.3639v6 [math.RT].
- [2] Y. Berest, P. Etingof, V. Ginzburg, *Finite dimensional representations of rational Cherednik algebras*, Int. Math. Res. Not. 2003, no. 19, 1053-1088, arXiv:math/0208138v3 [math.RT].
- [3] E. Briand, R. Orellana, M. Rosas, *The stability of the Kronecker product of Schur functions*, arXiv:0907.4652v2 [math.RT].
- [4] J. Comes, V. Ostrik, *On blocks of Deligne's category $\text{Rep}(S_t)$* , Advances in Mathematics 226 (2011), no. 2, 1331-1377, arXiv:0910.5695v2 [math.RT].
- [5] P. Deligne, *La Catégorie des Représentations du Groupe Symétrique S_t , lorsque t n'est pas un Entier Naturel*, <http://www.math.ias.edu/phares/deligne/preprints.html>.
- [6] R. Dipper, G.D. James, *Blocks and idempotents of Hecke algebras of general linear groups*, Proc. London Math. Soc. 54 (1987), 57-82.
- [7] P. Etingof, *Representation theory in complex rank I*, draft.
- [8] K. Erdmann, D. K. Nakano, *Representation type of Hecke algebras of type A*, Trans. Amer. Math. Soc. 354 (2002), 275-285.
- [9] P. Etingof, X. Ma, *Lecture notes on Cherednik algebras*, arXiv:1001.0432v4 [math.RT].
- [10] P. Etingof, E. Stoica, S. Griffeth, *Unitary representations of rational Cherednik algebras*, Representation Theory. 13 (2009): 349370, 2009 American Mathematical Society, arXiv:0901.4595v3 [math.RT].
- [11] W. Fulton, J. Harris, *Representation Theory: A First Course*, Graduate Texts in Mathematics Series, Springer-Verlag, 1991.
- [12] V. Ginzburg, N. Guay, E. Opdam, and R. Rouquier, *On the category O for rational Cherednik algebras*, Invent. Math. 154 (2003), no. 3, 617-651, arXiv:math/0212036v4 [math.RT].
- [13] E. Gorsky, A. Oblomkov, J. Rasmussen, V. Shende, *Torus knots and the rational DAHA*, arXiv:1207.4523v1 [math.RT].
- [14] I. Gordon, J. T. Stafford, *Rational Cherednik algebras and Hilbert schemes*, Advances in Mathematics, 198 (2005) 222-274, arXiv:math/0407516v2 [math.RA].
- [15] F. Knop, *A construction of semisimple tensor categories*, C. R. Acad. Sci. Paris, Ser. I 343 (2006), arXiv:math/0605126v2 [math.CT].
- [16] I. Losev *Towards multiplicities for categories O of cyclotomic rational Cherednik algebras*, arXiv:1207.1299v2 [math.RT].
- [17] I.G. Macdonald, *Symmetric Functions and Hall polynomials*, Oxford Mathematical Monographs, 1995.
- [18] A. Mathas, *Iwahori-Hecke Algebras and Schur Algebras of the Symmetric Group*, Univ. Lecture Series, Vol. 15, Amer. Math. Soc. (1999).
- [19] A. Mathew, *Categories parameterized by schemes and representation theory in complex rank*, arxiv:1006.1381v1 [math.RT].
- [20] R. Rouquier, *q -Schur Algebras and Complex Reflection Groups I*, Moscow Mathematical Journal 8 (2008), 119-158, arXiv:math/0509252v2 [math.RT].
- [21] R. Rouquier, *Representations of rational Cherednik algebras*, "Infinite-dimensional aspects of representation theory and applications", pp. 103-131, American Math. Soc., 2005, arXiv:math/0504600v2 [math.RT].

INNA ENTOVA AIZENBUD, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, DEPARTMENT OF MATHEMATICS, CAMBRIDGE, MA 02139 USA.

E-mail address: inna.entova@gmail.com